MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Ian Coley **Email/Phone:** msri@iancoley.org

Speaker's Name: Jacob Lurie

Talk Title: Character theory and tempered cohomology

Date: 3 / 28 / 19 **Time:** 9 : 30 **am** / pm (circle one)

Please summarize the lecture in 5 or fewer sentences:

Tempered cohomology is a way to study general cohomology theories via a more refined p-local method that "works better than it has any right to". They detail the construction and key applications.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

CHARACTER THEORY AND TEMPERED COHOMOLOGY

JACOB LURIE

To start, let's review classical representation theory. Let G be a finite group. Then we get $\mathrm{Rep}(G)$ the representation ring. There's a map

$$\mathrm{Rep}(G) \to \{\text{class functions } f\colon G \to \mathbb{C}\}$$

and $\mathbb{C} \otimes \mathrm{Reg}(G) \cong \{\text{class functions}\}$ via $[V] \mapsto \chi_V$.

Now if G is a finite group acting on a space X (like a finite G-CW-complex), the equivariant Chern character gives

$$\mathrm{ch}_G\colon \mathbb{C} \otimes K^0_G(X) \xrightarrow{\cong} H^{\mathrm{even}}(\{(g,x) : g(x) = x\}/G; \mathbb{C})$$

where G-equivariant complex K-theory is the Grothendieck group of G-equivariant $\mathbb{C}$-vector bundles on X and the G action on the subset of $G \times X$ is by conjugation on G and the action on X.

Example 1. Examples:

- $G = *$, then we recover $\mathrm{ch}\colon K^0(X) \otimes \mathbb{C} \xrightarrow{\cong} H^{\mathrm{even}}(X; \mathbb{C})$, and this works over $\mathbb{Q}$ too.
- If $X = *$, then we get back the classical story, which does need to be done over $\mathbb{C}$.

And now a chromatic interpretation: this is a comparison between two cohomology theories. K^0 is height 1, but $H^{\mathrm{even}}(-; \mathbb{C})$ is height 0. So after complexifying we remember a bit of the height 1 information as height 0 information on a more complicated space.

Chromatic interpretation continued:

Let k be a perfect field of characteristic p. Let $\mathbb{G}_0 \to \mathrm{Spec}\, k$ be a formal group of dimension 1 of height $n < \infty$. Lubin-Tate gives us a universal deformation $\mathbb{G}$ of $\mathbb{G}_0$ over the Lubin-Tate ring (k-algebra) R, where R looks like $W(k)[[v_1, \ldots, v_{n-1}]]$.

Notes by Ian Coley.

Morava observed that this sort of thing can happen in homotopy theory too. There exist a unique cohomology theory E such that $E^0(*) = R$ and $E^0(\mathbb{CP}^\infty)$ is the ring of functions on $\mathbb{G}$. This is Morava E-theory.

Example 2. Let $k = \mathbb{F}_p$, $\mathbb{G}_0 = \mathbb{G}_m$ the formal multiplicative group. Then $R = \mathbb{Z}_p$ and $E = (K^0)_p^\wedge$ the p-adic completion of complex K-theory a "higher height version". The higher the height, the more information E sees.

Hopkins-Kuhn-Ravenel compares height n theories with height 0 theories. Let G act on a finite G-CW complex X. Then $E^0(X_{hG})$ is a R-algebra, where E^0 is Borel-equivariant E-theory. But if R embeds into $\mathcal{C}$, we could also complexify. This happens if (e.g.) k is a finite field. Better, we can embed $R[1/p]$ into a characteristic zero L, a more natural extension, where L classifies trivializations of $T(\mathbb{G})$ the Tate module.

Theorem 3 (HKR)**.**

$$L \otimes_R E^0(X_{hG}) \cong H^{\text{even}}(\{(g_1, \ldots, g_n, x) \in G^n \times X : \\ g_i(x) = x, g_i g_j = g_j g_i, g_i^{p^k} = 1 \text{ for big } k\}/G; L)$$

where we think of the left side as "torsion" and the right side as "rational".

There's a similar story to go from height n to lower heights (not just zero) by Stapleton.

Time for DAG:

Definition 4 (Tate)**.** Let R be a commutative ring, p a prime, A a *p-divisible group over* R is a functor $\mathbb{G}\colon \mathbf{CAlg}_R \to \mathbf{Ab}$ such that:

(1) $\mathbb{G}(A)$ is a p-power torsion abelian group for all $A \in \mathbf{CAlg}_R$
(2) For all $m \geqslant 0$, $\mathbb{G}[p^m] :: A \mapsto \mathbb{G}(A)[p^m]$ the p^m-torsion elements is representable by a finite flat group scheme over R
(3) $p\colon \mathbb{G} \to \mathbb{G}$ is surjective (locally as a sheaf in the flat topology)

Example 5. $\mathbb{G} = \mu_{p^\infty}$, where $\mu_{p^\infty}(A) = \{x \in A : x^{p^k} = 1 \text{ for } k >> 0\}$. In this case, (1) is obvious, (2) holds because $\mu_{p^\infty}[p^m] = \mu_{p^m}$ is represented by $\mathrm{Spec}(R[t]/(t^{p^m} - 1) = \mathrm{Spec}(R[\mathbb{Z}/p^m])$ which is finite flat. For (3), if $x \in A$, $A[t]/(t^p - x)$ is a faithfully flat extension of A and it turns x into a pth power.

Now, let R be an $\mathbb{E}_\infty$-ring spectrum.

Definition 6. A p-divisible group $\mathbb{G}$ over R is $\mathbb{G}\colon \mathbb{E}_\infty\textbf{-Alg}_R \to \textbf{sAb}$ valued in simplicial abelian groups satisfying analogous axioms, as demonstrated by example:

Example 7. Same example! $\mu_{p^\infty} = \operatorname{colim} \mu_{p^m}$, where $\mu_{p^m}(A) = \operatorname{Hom}_{\mathrm{Sp}}(H\mathbb{Z}/p^m, \operatorname{gl}_1 A)$, or, μ_{p^m} is represented by $\operatorname{Spec}(R \wedge (\mathbb{Z}/p^m)_+)$. It pretty clearly satisfies all the analogous properties.

In rings, if H is a finite abelian group, then let $\widehat{H} = \operatorname{Hom}_{\mathbf{Ab}}(H, \mathbb{C}^\times)$. Then $\operatorname{Rep}(H) = \mathbb{Z}[\widehat{H}]$ because H is abelian.

How do we upgrade this to spectra? $\operatorname{Rep}(H)$ feels a lot like H-equivariant complex K-theory, $K^H \simeq K[\widehat{H}]$. But in order to do this, we don't really need H, just BH, as K^H is actually the K-theory of finite dimension $\mathbb{C}$-local systems on BH. So we can get at the group algebra $K[\widehat{H}]$ only using BH and, in particular, without knowing $\widehat{H}$.

Definition 8. Let R be an $\mathbb{E}_\infty$-ring spectrum, and $\mathbb{G}$ a p-divisible group over R.

(Version 1) A *preorientation* on $\mathbb{G}$ is $R_{\mathbb{G}}\colon \mathcal{T}^{\mathrm{op}} \to \mathbb{E}_\infty\textbf{-Alg}_R$, $T \mapsto R_{\mathbb{G}}^T$, such that for H a finite abelian p-group, $\operatorname{Spec}(R_{\mathbb{G}}^{BH}) = \operatorname{Hom}(\widehat{H}, \mathbb{G})$, where $\mathcal{T}$ is the category of spaces isomorphic to BH for H a finite abelian p-group, which we think of a topological/∞-category.

This of this as an axiomatic for "representation ring on abelian groups are group algebras". So the data here is really in the morphisms, because it we know what it does on all objects.

(Version 2) A *preorienation* on $\mathbb{G}$ is a map of simplicial abelian groups $(\mathbb{Q}_p/\mathbb{Z}_p)[1] \to \mathbb{G}(R)$.

There are also other definitions.

Note: for $T \in \mathcal{T}$, each point $t \in T$ gives us a map $R_{\mathbb{G}}^T \to R_{\mathbb{G}}^{\{t\}} = R$ and these assemble into a comparison map $\zeta\colon R_{\mathbb{G}}^T \to R^T = \operatorname{Map}(T, R)$.

Example 9. Let R be complex K-theory, $\mathbb{G} = \mu_{p^\infty}$, then $\pi_0\zeta\colon \operatorname{Rep}(H) \to K^0(BH)$.

Theorem 10 (Atiyah)**.** The righthand side is the completion of the lefthand side with respect to the augmentation ideal.

Definition 11. A preorientation is an *orientation* if it satisfies Atiyah's theorem, not just on π_0.

Construction:

Let X be any space. Then we get a comparison

$$\zeta\colon R_{\mathbb{G}}^X := \operatorname*{holim}_{T\to X, T\in\mathcal{T}} R_{\mathbb{G}}^T \to \operatorname*{holim}_{T\to X, T\simeq *}(R_{\mathbb{G}}^T = R) = R^X$$

Let $R_{\mathbb{G}}^*(X) = \pi_* R_{\mathbb{G}}^X$ be $\mathbb{G}$-*tempered cohomology*, which compares to $R^*(X) = \pi_* R^X$ plain old R-*cohomology*. So the lefthand side goes to the righthand side by competing ... in some way.

So why even talk about this?

(Property 1) If $\mathbb{G}$ is connected, then the comparison map is an isomorphism, and we can use $\mathbb{G}[p^m] = \operatorname{Spec}(R^{B\mathbb{Z}/p^m})$ to recover $\mathbb{G}$. So we've not done much.

(Property 2) Suppose $\mathbb{G} = \mathbb{G}_0 \oplus \underline{\mathbb{Q}_p/\mathbb{Z}_p}$, something plus a constant group. Then

$$R_{\mathbb{G}}^*(X) \cong R_{\mathbb{G}_0}^*(X^{B\mathbb{Z}_p})$$

which (apparently) is not a difficult calculation.

(Property 3) Let $\mathbb{G}$ be an oriented p-divisible group over A, $f\colon A \to B$. Then as spectra we get $A \otimes_A A_{\mathbb{G}}^X \simeq B_{\mathbb{G}}^X$, a map of B-algebras, whenever X is π-finite, i.e. it has only finitely many homotopy groups, and those themselves are finite. The proof is not easy.

Example 12. Let $A = KU$, $\mathbb{G} = \bigoplus_p \mu_{p^\infty}$, $B = \mathbb{C} \otimes_{\mathbb{S}} KU$, $X = BG$, for G a finite group. Then Property 3 on π_0 says that $\mathbb{C} \otimes_{\mathbb{Z}} \operatorname{Rep}(G) \cong \{\text{class functions } G \to \mathbb{C}\}$.

Proof. (of Property 3) Categorification of all of the above: In some "tempered local systems category", we have

$$\operatorname{Ext}^* = A_{\mathbb{G}}^*(X) \xrightarrow{\zeta} A^*(X) = \operatorname{Ext}^*(\underline{A}_X, \underline{A}_X)$$

where the latter Ext is in some other stable ∞-cat.

The map ζ arises as some sort of ambidexterity phenomenon in the case of π-finite X and a comparison of Ext groups. □

Remark 13. This ambidexterity showed up in $K(n)$-local spectra in work with Hopkins, but it can be found without that sacrifice if you change your viewpoint to tempered local systems.

Additionally, p-divisible groups show up as torsion points of (oriented) elliptic curves. If you put that in, elliptic equivariant cohomology pops out.

Lastly, for X π-finite and E Morava E-theory, $\mathbb{G}$ an associative p-divisible group over E and connected, then

$$\begin{aligned}
\text{(Property 1)} \quad L \otimes_R E^0(X) &= L \otimes_R E^0_{\mathbb{G}}(X) \\
&= \pi_0(H^{\text{even}}(L \otimes_E E^X_{\mathbb{G}})) \\
\text{(Property 3)} &= \pi_0(H^{\text{even}}(L^X_{\mathbb{G}})) \\
&= H^{\text{even}}_{\mathbb{G}}(X; L)
\end{aligned}$$

but L is designed so that $\mathbb{G}$ is trivialized, thus

$$\begin{aligned}
&= H^{\text{even}}_{(\mathbb{Q}_p/\mathbb{Z}_p)^n}(X; L) \\
\text{(Property } 2^n\text{)} &= H^{\text{even}}(X^{B(\mathbb{Z}_p)^n}; L) \\
&= \prod_{[B(\mathbb{Z}_p)^n \to X]} L
\end{aligned}$$

This also allows us to "unextended scalars":

$$E^0(X)[1/p] = \left(\prod_{[B(\mathbb{Z}_p)^n \to X]} L \right)^{\mathrm{GL}_n(\mathbb{Z}_p)}$$