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NOTETAKER CHECKLIST FORM

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Name: Ian Coley **Email/Phone:** msri@iancoley.org

Speaker's Name: Akhil Mathew

Talk Title: p-adic algebraic K-theory and topological cyclic homology

Date: 3 / 27 / 19 **Time:** 9 : 30 [✓ am] / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: Older and recent work has shown how TC is useful in computing algebraic K-theory. They show hot to piece together l-adic and p-adic results into a general new invariant called Selmer K-theory.

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p-ADIC ALGEBRAIC K-THEORY AND TOPOLOGICAL CYCLIC HOMOLOGY

AKHIL MATHEW

Joint with Clausen, based on work of Clausen-Mathew-Morrow and Clausen-Mathew-Naumann-Noel.

I. Motivation for K-theory

Let R be a commutative ring. One can associate a family of groups $K_i(R)$ such that $K_0(R)$ is the Grothendieck group of finitely generated projective R-modules, and the higher K-groups come as homotopy groups of some space $K(R)$

$K_0(R)$ is a literal group completion of isomorphism classes of finitely generated projective R-modules, so one way to see the space $K(R)$ is as a homotopy-theoretic group completion of the *category* of finitely generated projective R-modules.

Definition 1. (Definition/Construction) Given a stable ∞-category $\mathcal{C}$ (e.g. a dg category), there's a general construction of a spectrum $K(\mathcal{C})$. In particular, $\pi_0 K(\mathcal{C})$ is the free abelian group on $x \in \mathcal{C}$ modulo cofibre sequences: if $x' \to x \to x''$ is a cofibre sequence, this $[x] = [x'] + [x'']$. This has a precise universal property [Blumberg-Gepner-Tabuada].

So given a quasicompact quasiseparated scheme X, define $K(X) := K(\mathrm{Perf}(X))$. If $X = \operatorname{Spec} R$, it's the same thing.

Theorem 2 (Thomason-Trobaugh)**.** $X \mapsto X(K)$ is a sheaf for the Nisnevich topology.

It's difficult to compute in general, so most computations are done obliquely rather than from the definition. New computations come usually from new (discovered) properties. So: think of $X \mapsto K(X)$ like a cohomology theory on schemes, which can be made more explicit via étale, prismatic, crystalline, etc.

Notes by Ian Coley.

But: algebraic K-theory depends only on $\mathrm{Perf}(X)$, as opposed to the other cohomology theories – it's a *noncommutative* invariant.

Example 3. If $f\colon Y \to X$ is smooth and proper, $\mathbb{R}f_*\colon \mathrm{Perf}(Y) \to \mathrm{Perf}(X)$ induces $f_*\colon K(Y) \to K(X)$. In general, this doesn't come as freely (if at all).

Example 4 (Thomason)**.** Grothendieck's absolute purity conjecture: a Gysin sequence in ℓ-adic cohomology. The analogue in K-theory is dévissage.

Example 5 (Voevodsky)**.** If $X/\mathbb{C}$ is a variety, for each Zariski open $U \subset X$, consider $H^*(U, \mathbb{Z}/\ell)$. Then $\mathrm{colim}_U\, H^*(U, \mathbb{Z}/\ell)$ is generated in degree 1. This is a special case of Bloch-Kato for $\mathbb{C}(X)$.

Focus: algebraic K-theory with torsion coefficients or profinitely completed $K^\wedge$. The difficulty is that this is not a sheaf for the étale topology, so you can't use Galois theory to compute $K(k)$ from its separate extensions.

The principle is that there is an approximation to K-theory that does satisfy étale descent and is easier to compute. We could just étale sheafify, but that loses some lovely features, like only depending on $\mathrm{Perf}(X)$.

II. ℓ-adic K-theory

Theorem 6 (Suslin-Gabber)**.** If k is a separately closed field of characteristic $\neq \ell$, then $K(k)^\wedge_\ell \cong ku^\wedge_\ell$, noncanonically, and $\pi_{2i}K(k)^\wedge_\ell \cong \mathbb{Z}_\ell(i)$. The same holds for rings R which are strictly henselian with residue field k.

New principle: $\mathbb{A}^1$-homotopy invariance for smooth algebras.

From now on, everything is ℓ-adically complete, $\ell > 2$.

Construction 7 (Miller-Mahowald)**.** There is a functor $L_{K(1)}\colon \mathrm{Sp} \to \mathrm{Sp}$. Explicitly, if $X \in \mathrm{Sp}$, then $L_{K(1)}(X)/\ell = X/\ell[v_1^{-1}]$ where $v_1\colon \Sigma^{2\ell-2}(S^0/\ell) \to S^0/\ell$ is something purely stable homotopy theoretic.

Theorem 8 (Thomason)**.** Let X be a scheme over $\mathbb{Z}[1/\ell]$ plus some finiteness. Then $L_{K(1)}K(-)$ satisfies étale descent. Moreover, there is a spectral sequence $H^s_{\text{ét}}(X, \mathbb{Z}_\ell(t)) \implies \pi_{2t-s}L_{K(1)}K(X)$.

Example 9. If $X/\mathbb{C}$ is a variety, $L_{K(1)}K(X) = KU^\wedge_\ell(X(\mathbb{C}))$.

By our principle, the last thing to ask is: is this a good approximation? Yes! Consider the map $\star\colon K(x)^\wedge_\ell \to L_{K(1)}K(X)$

Theorem 10 (Rosenschon-Østvaer, after Voevodsky-Rost)**.** Suppose X has finite Krull dimension and for all $x \in X$, $\mathrm{vcd}_\ell(k(x)) \leqslant d$. Then $\star$ is an isomorphism on π_* with $* \geqslant \max(d-2, 0)$.

III. p-adic K-theory

Suslin-Gabber no longer applies – K-theory is not locally constant!

Example 11. New examples:

- $K(\overline{\mathbb{F}}_p)^\wedge_p \simeq H\mathbb{Z}_p$, proven by Quillen, concentrated on degree zero.
- $K(\mathcal{O}_{\mathbb{C}_p})^\wedge_p \simeq ku^\wedge_p$, proven by Niziol, so somehow this is the same as the ℓ-adic case.

One complication is that we are no longer insensitive to nil-ideals. Our new friend is topological cyclic homology TC.

Construction 12 (Bökstedt-Hsiang-Madsen '93)**.** Given a stable ∞-category $\mathcal{C}$, one constructs a spectrum and a map $K(\mathcal{C}) \to TC(\mathcal{C})$. The definition of TC is more complicated, but the computations are easier. It's derived from THH, which is derived from regular HH.

Theorem 13 (Dundas-Goodwillie-McCarthy)**.** If (R, I) is a ring with a nilpotent ideal, then there exists a homotopy cartesian square

$$\begin{tikzcd}
K(R) \arrow[r] \arrow[d] & K(R/I) \arrow[d] \\
TC(R) \arrow[r] & TC(R/I)
\end{tikzcd}$$

Thus relative K-theory agrees with relative TC. This is also explored by Hesselholt-Madsen (etc) to do many computations.

Does this fit our principle? Let R be a p-complete ring. p-adically, $K(R)^\wedge_p \to TC(R)^\wedge_p$ is a good approximation.

Theorem 14 (Geissar-Hesselholt, CMM)**.** If R is p-complete, $TC(R)^\wedge_p$ is étale p-adic K-theory in degrees $\geqslant 0$. Also, $K^\wedge_p \to TC^\wedge_p$ is an equivalence in large enough degrees (on rings that aren't too large).

IV. Gluing things together

Definition 15 (Clausen)**.** For $\mathcal{C}$ a stable ∞-category, then define the Selmer K-theory $K^{\mathrm{Sel}}(\mathcal{C}) := L_1 K(\mathcal{C}) \times_{L_1 TC(\mathcal{C})} TC(\mathcal{C})$, where L_1 is like $L_{K(1)}$.

Example 16. This glues up II and III nicely.

- $K^{\mathrm{Sel}}(\mathcal{C})_{\mathbb{Q}} \simeq K(\mathcal{C})_{\mathbb{Q}}$
- $K^{\mathrm{Sel}}(\mathcal{C})^{\wedge}_{\ell} \simeq L_{K(1)} K(\mathcal{C})$ if ℓ is invertible on $\mathcal{C}$
- $K^{\mathrm{Sel}}(R)^{\wedge}_{p} \simeq TC(R)^{\wedge}_{p}$ if R is p-complete

So it looks great, but does it fit our principle? There exists a map $K \to K^{\mathrm{Sel}}$ in general, but is it any good? Further, K^{Sel} is purely categorical, but is it an étale sheaf?

Theorem 17 (CM,CMNN)**.** The construction $X \mapsto K^{\mathrm{Sel}}(X)$ is an étale sheaf on spectral schemes.

Theorem 18 (CM)**.** $K^{\text{ét}} \to K^{\mathrm{Sel}}$ is an equivalence in degrees $\geqslant -1$.

So: étale K-theory is (up to negative business) a noncommutative invariant.

Definition 19. Let k be a field, p a prime. Let d_k be $\mathrm{vcd}_p(k)$ if the characteristic of k is not p, and let d_k be $1 + \log_p[k : k^p]$ otherwise.

Theorem 20 (CM)**.** If X is a quasicompact quasiseparated spectral scheme of finite Krull dimension and $d = \sup_{x \in X} d_{k(x)}$, then $K(X)^{\wedge}_{p} \to K^{\mathrm{Sel}}(X)^{\wedge}_{p}$ is an isomorphism in degrees $\geqslant \max(d-2, 0)$.

Ingredients: a technical issue called 'hypercompleteness' on étale sheaves of spectra; enhancement of Dundas-Goodwillie-McCarthy to spectral stuff.