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Wall structures in mirror symmetry

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I. A-model walls

In mirror symm,

A-model (X)	B-model $(\check{X})$
symplectic (Kähler)	complex

SYZ: (cf Denis' talk)

$$\begin{tikzcd} X \arrow[dr, "\pi"'] & & \check{X} = TB_0/\Lambda \arrow[dl, "\pi^\vee"] \\ & B & \end{tikzcd}$$

$B_0 = B \setminus \Delta \subset B$

then $X = T^*B_0/\Lambda^*$

Λ : $\mathbb{Z}$-tang. vec.

$\mathbb{Z}$-affine mfld

sing. locus $\Delta \subset B$

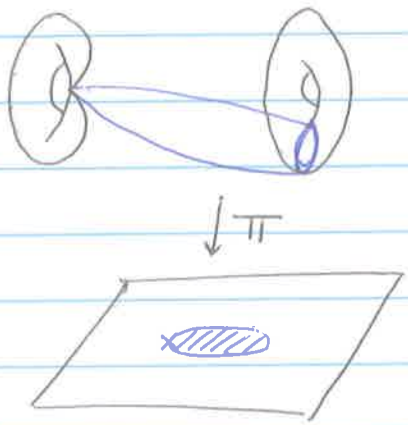
$\operatorname{codim}_{\mathbb{R}} \Delta = 2$

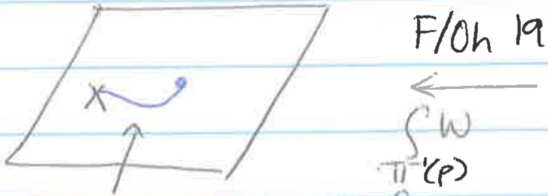
Pblm: How to correct the complex structure on TB_0/Λ to get $\check{X}$?

Fukaya 2001: Multivalued Morse theory

Corrections for $\check{X}$ from (ps-) holomorphic disks $\subset X$

Near singular fibre:





gradient line for $f = \int_D \omega$

II. B-model walls : Initial walls

Gross / Siebert : study SYZ via toric degenerations

$$\begin{tikzcd}
X \arrow[d] & \mathcal{X} \arrow[d] & \supset & X_0 \arrow[d] & \check{X}_0 \arrow[d] & \in & \check{\mathcal{X}} \arrow[d] & \check{X} \arrow[d] \\
S & S & \ni & 0 & 0 & \in & \check{S} & \check{S}
\end{tikzcd}$$

central elts

$X_0, \check{X}_0 = \bigcup$ toric ↔ momentum polytops σ

$B = \bigcup \sigma$

has perfect mirror duality $(X_0, \text{log-str}, \text{polariz}) \longleftrightarrow (\check{X}_0, \dots)$

Reconstruction :

Given X_0, how to construct $\mathcal{X}$?

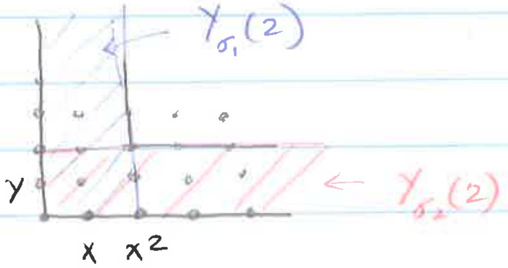
Log-smooth deformation : $n=2$ ✓ $n = \dim \mathcal{X}$

$n>2$: doesn't work

Instead : $\mathcal{X}/t^{k+1}$ $(S = \operatorname{Spec} \mathbb{C}[[t]])$

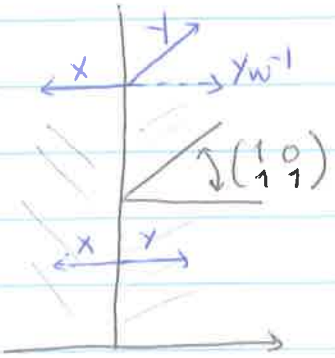
primary decomposition $\mathcal{X} \backslash t^{k+1} = \bigcup_\sigma Y_\sigma(k)$, $Y_\sigma(0)$ toric

e.g. $n=1$: $xy=t$
mod t^2



B :

$n=2$



B →

toric suggestions:

← $xy = wt$

↕ don't agree

← $xy = t$

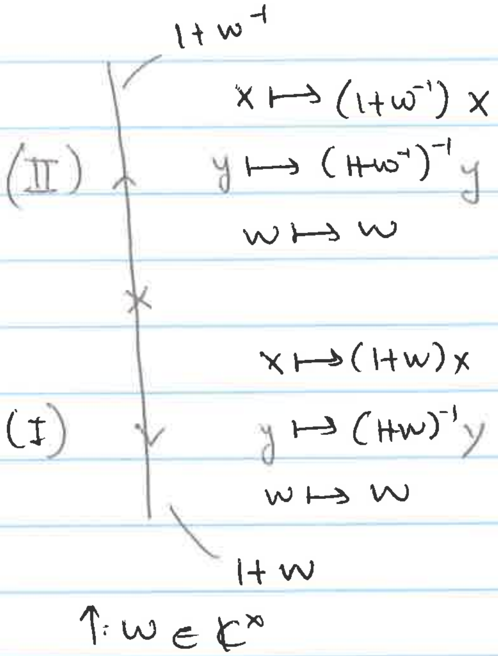
generators of glued rings:

(I) $X = (x, (1+w)x)$

$Y = ((1+w)y, y)$

$W = (w, w)$

$XY = (1+w).t$



(II) $X = (x, (1+w^{-1})x)$

$Y = ((1+w^{-1})y, y)$

$W = (w, w)$

$$XY = ((1+w^{-1})\underset{\overset{\shortparallel}{wt}}{xy}, (1+w^{-1})xy) = (1+w).t$$

Ex: $\mathcal{X} = \{ z_0 \cdots z_3 + t \sum z_i^4 = 0 \} \subset \mathbb{P}^3 \times \mathbb{C}$

$X_0 = \bigcup \mathbb{P}^2$, $B =$

Δ

Upshot: ~~lin~~ Initial walls $\subset$ $(n-1)$-skeleton of $\{\sigma\}$ provide local (non-toric) models of $\mathcal{X}$ near $\Delta \subset B$

III B-models: Scattering

Introducing corrections to patching of $\mathcal{X}$ locally yields problems globally.

Kontsevich - Soibelman (2004) For K3 $(n=2)$ construct $\mathcal{X}$ as a rigid analytic space by interpreting Fukaya's gradient lines as carrying corrections to patching standard rigin analy. charts for $\mathcal{X}$.

group ruling patching : pro-unipotent alg. group

$$G = \varprojlim_{\lambda \in \mathbb{R}} G_\lambda$$

$\Rightarrow$ $\exists$ unique way of introducing new gradient lines + $\overset{\text{(hol.)}}{\text{sympl.}}$ at intersections.

Gross. Siebert '07: Given B, (with data), $\exists$ a canonical smoothing $\mathfrak{X}$ of $X_0 = \bigcup$ tori var(s) via a wall structure on B.

<u>group</u> : $\subset \mathrm{Aut}_{\mathbb{C}[[t]]} \mathbb{C}[[t]][M]$, $M \simeq \mathbb{Z}^n$

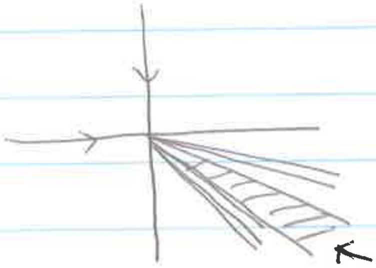
Lie alg gen $z^m \partial_n$, $\langle m, n \rangle = 0$ $n \in N = M^*$.
walls $\subset n^\perp$

Compare to cluster world : $z^{p^*(n)} \partial_n$, $p^* : N \to M$

<u>Important</u> : work on the X-side making the corrections by disks from the mirror side tropical.

Th^m (GHKS) : $\mathfrak{X}$ comes with a canonical basis of sections of $\mathcal{O}_{\mathfrak{X}}(d)$ labelled by $B(\frac{1}{d}\mathbb{Z})$

Note : Also in cluster world, the degeneration makes it possible to look into "bad lands"



IV B-model walls(X) = A-model wall $(\check{X})$

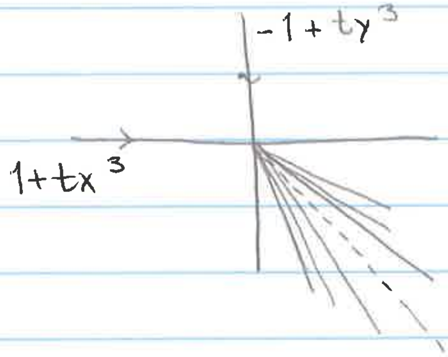
Fundamental case: (Gross - Pandharipande - Siebert 2009):

"the tropical vertex" Scattering in $n=2$

$\Updownarrow$

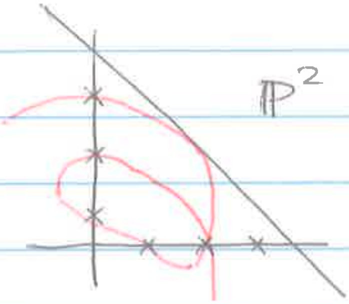
(relative) Gromov-Witten theory on toric varieties

Ex.



$\leftarrow \exp\left(\underline{\underline{9}}t^2xy + 2\cdot\frac{63}{\underline{\underline{4}}}(t^3xy)^2 + 3\cdot\underline{\underline{55}}(t^3xy)^3 + \cdots\right)$

$55 = 1 + 18 + 12\cdot\underline{\underline{3}}$



Rem: This allows to reverse the logic and construct wall structures from mirrors of (Y,D) log-CY surface

$\overset{\wr}{\text{smoothing } \mathcal{X}}$

In general, one ~~me~~ needs punctured (logarithmic) invariants of (X_0, μ_{X_0}) to interpret the walls

$\leftarrow$ log struct

in progress

Abramovich / Chen / G/S