PROPER AFFINE ACTIONS OF RIGHT-ANGLED COXETER GROUPS

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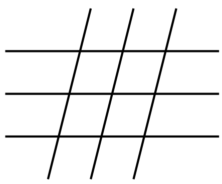
ABSTRACT. The Auslander Conjecture states that all discrete groups acting properly and cocompactly on $\mathbb{R}^n$ by affine transformations should be virtually solvable. In 1983, Margulis constructed the first examples of proper (but not cocompact) affine actions of non-abelian free groups. It seems that until now all known examples of irreducible proper affine actions were by virtually solvable or virtually free groups. I will explain that any right-angled Coxeter group on k generators admits a proper affine action on $\mathbb{R}^{k(k-1)/2}$. This is joint work with J. Danciger and F. Guéritaud.

Question. Understand proper affine actions of f.g. $\Gamma \underset{\tau}{\curvearrowright} \mathbb{R}^N$, $\tau : \Gamma \to \mathrm{Aff}(\mathbb{R}^N) = GL_N(\mathbb{R}) \ltimes \mathbb{R}^N$ faithful.

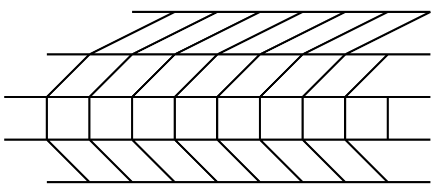
Proper $\iff$ $\tau(\Gamma)\backslash\mathbb{R}^N$ manifold (orbifold) $\iff$ $\tau(\Gamma)$ symmetry group of periodic affine tiling of $\mathbb{R}^n$ (tiles possibly noncompact).

Examples.

(1) $\mathbb{Z}^N \curvearrowright \mathbb{R}^N$ by translations.



(2) $\langle a, b\rangle = \mathbb{Z}^2 \curvearrowright \mathbb{R}^2$.



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$$\tau : a \mapsto \text{translation by} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$b \mapsto \left(x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

(3) $\mathbb{Z}^2 \rtimes \mathbb{Z} \curvearrowright \mathbb{R}^3$, where $\langle a, b \rangle = \mathbb{Z}^2$ and $\langle c \rangle = \mathbb{Z}$ factor.

$$\tau : a \mapsto \text{translation by} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$b \mapsto \text{translation by} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$c \mapsto \left(x \mapsto \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array} \right) x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$A \in SL_2\mathbb{Z}$ with three cases:

- identity, group $\mathbb{Z}^2 \rtimes \mathbb{Z} \cong \mathbb{Z}^3$ acting by translations
- parabolic/elliptic, group is a Heisenberg group
- hyperbolic, group is solvable but not nilpotent

Conjecture (Auslander, 1964). $\Gamma \underset{\tau}{\curvearrowright} \mathbb{R}^N$ proper, $\tau(\Gamma)\backslash\mathbb{R}^N$ compact

$\implies \Gamma$ is virtually polycyclic.

Case $\tau(\Gamma) \subset O(N) \ltimes \mathbb{R}^N$ ("crystallographic"): Γ is virtually $\mathbb{Z}^N$ acting by translations (Bieberbach, 1911)

The Auslander Conjecture has been proved for $n \leq 6$

- 2: easy
- 3: Fried–Goldman
- 4,5,6: Abels–Margulis–Soifer

Milnor (1977): is the conjecture true if $\tau(\Gamma)\backslash\mathbb{R}^N$ noncompact?

Margulis (1983): NO! There exist proper affine actions of the free group $\mathbb{F}_r$ on $\mathbb{R}^3$ for all $r \geq 2$.

$\tau(\Gamma) \subset O(2,1) \ltimes \mathbb{R}^3$

$\tau(\Gamma)\backslash\mathbb{R}^3$ "Margulis spacetime" (flat Lorentzian manifold)

In dimension 3, either Γ is virtually polycyclic, or it is virtually free and we get a Margulis spacetime. What about higher dimension?

- Γ virtually polycyclic.
- Γ virtually free (Abels–Margulis–Soifer, Goldman–Labourie–Margulis, Smilga).
- What about other examples?

Theorem 1. *Any right-angled Coxeter group in k generators admits proper affine actions on* $\mathbb{R}^{k(k-1)/2}$.

Corollary. *Any group commensurable to a subgroup of a RACG admits proper affine actions.*

Examples.

- all RAAGs (Davis–Januszkiewicz)
- all virtually special groups (Haglund–Wise)
- all Coxeter groups (Haglund–Wise)
- all hyperbolic cubulated groups (Agol), including for example $\pi_1($ closed hyperbolic 3-manifold$)$ (Kahn–Markovic + Sageev)

I. General Setting

G Lie group $(G \times G) \curvearrowright G$: $(g_1, g_2) \cdot g = g_2 g g_1^{-1}$	$\mathfrak{g}$ Lie algebra $(G \ltimes \mathfrak{g}) \curvearrowright \mathfrak{g}$ affine: $(g, w) \cdot v = \mathrm{Ad}(g)v + w$

Γ discrete group

$\begin{cases} \rho \ : \Gamma \to G & \text{group hom.} \\ \rho' : \Gamma \to G & \text{group hom.} \end{cases}$	$\begin{cases} \rho : \Gamma \to G & \text{group hom.} \\ u : \Gamma \to \mathfrak{g} & \rho\text{-cocycle :} \end{cases}$ $u(\gamma_1\gamma_2) = u(\gamma_1) + \mathrm{Ad}\,\rho(\gamma_1)u(\gamma_2)$

NB:

$$T_\rho \operatorname{Hom}(\Gamma, G) \hookrightarrow \{\rho\text{-cocycles } u : \Gamma \to \mathfrak{g}\}$$
$$\frac{d}{dt}\Big|_{t=0} \rho_t \mapsto u \text{ s.t. } \rho_t(\gamma) = e^{tu(\gamma)+o(t)}\rho(\gamma)\ \forall\, \gamma$$

with $\rho_0 = \rho$.

II. Principle: "uniform contraction $\implies$ properness"

(Notetaker's note: the following red text is added later.)

$G = O(n, 1)$ $\qquad$ $Q(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$

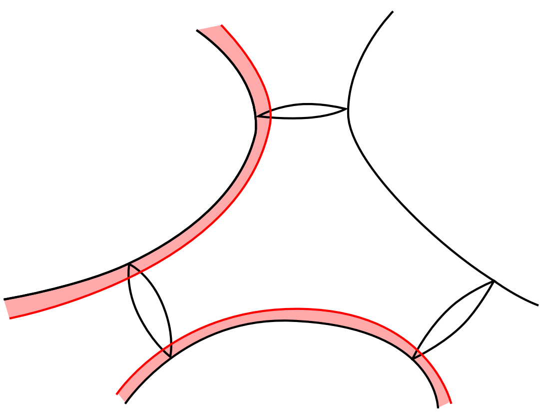
$O(p, q+1)$ $\qquad$ $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$

acting on $\mathbb{H}^n = \{[x] \in \mathbb{P}(\mathbb{R}^{n+1}) \mid Q(x) < 0\}$.

$\mathbb{H}^{p,q}$ $\qquad$ $\mathbb{R}^{p+q+1}$

Theorem 2. *$\rho : \Gamma \to G$ injective and discrete, preserving a properly convex open domain $\Omega \subset \mathbb{H}^{p,q}$.*

$\rho' : \Gamma \to G$ unif. contracting w.r.t. ρ in spacelike directions, and preserves properly convex domain $\Omega' \subset \mathbb{H}^{p,q}$ (technical assumption: $\rho'(\Gamma)$ Zariski dense)	*$u : \Gamma \to \mathfrak{g}$ unif. contracting in spacelike directions*
$\implies \Gamma \underset{(\rho,\rho')}{\curvearrowright} G$ *proper*	$\implies \Gamma \underset{(\rho,u)}{\curvearrowright} \mathfrak{g} \simeq \mathbb{R}^{n(n+1)/2}$ *proper* $\mathbb{R}^{(p+q)(p+q+1)/2}$



Definition. ρ' is *uniformly contracting with respect to* ρ in spacelike directions if $\exists f : \mathbb{H}^n \to \mathbb{H}^n \quad \Omega \to \Omega'$

- (ρ, ρ')-equivariant:

$$f(\rho(\gamma)z) = \rho'(\gamma) \cdot f(z)$$

- $\exists C < 1$ such that $\forall y, z \in \mathbb{H}^n$ Ω with $[y, z]$ spacelike,

$$d(f(y), f(z)) \leq Cd(y, z)$$

Definition. u is *uniformly contracting* in spacelike directions if $\exists X :$ $\mathbb{H}^n \to T\mathbb{H}^n \quad \Omega \to T\Omega$

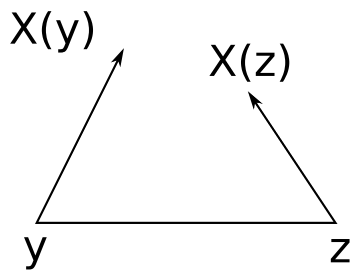
- (ρ, u)-equivariant:

$$X(\rho(\gamma) \cdot z) = \rho(\gamma)_* X(z) + u(\gamma)(\rho(\gamma) \cdot z)$$

(where $u(\gamma)(\rho(\gamma) \cdot z)$ means $\frac{d}{dt}|_{t=0} e^{tu(\gamma)} \rho(\gamma) \cdot z$)

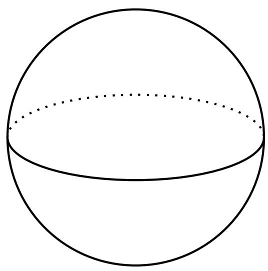
- $\exists c < 0$ such that $\forall y, z \in \mathbb{H}^n$ Ω with $[y, z]$ spacelike,

$$\frac{d}{dt}|_{t=0} d(exp_y(tX(y)), exp_z(tX(z))) \leq cd(y, z)$$

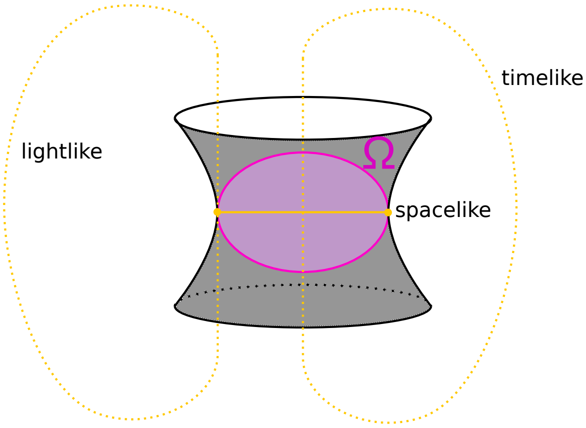


$\sim$ addition of red text to set up of Theorem 2 above begins $\sim$

$\mathbb{H}^{3,0} = \mathbb{H}^3 \subset \mathbb{P}(\mathbb{R}^4)$



$\mathbb{H}^{2,1} \subset \mathbb{P}(\mathbb{R}^4)$



affine chart $x_4 = 1$

Three types of geodesic: spacelike, timelike, lightlike.

Proof for $G = O(n,1)$.

$$\pi : O(n,1) \to \mathbb{H}^n$$
$$g \mapsto \text{unique fixed point of } g^{-1} \circ f$$

$$\pi : \mathfrak{o}(n,1) \to \mathbb{H}^n$$
$$v \mapsto \text{unique zero of } X - v$$

is well-defined, continuous, and equivariant w.r.t.

$$\Gamma \underset{(\rho,\rho')}{\curvearrowright} O(n,1) \text{ and } \Gamma \underset{\rho}{\curvearrowright} \mathbb{H}^n\text{, and respectively}$$
$$\Gamma \underset{(\rho,u)}{\curvearrowright} \mathfrak{o}(n,1) \text{ and } \Gamma \underset{\rho}{\curvearrowright} \mathbb{H}^n.$$

Action is proper on target $\implies$ proper on source. □

III. Proper actions of RACG

$$\Gamma = \langle \gamma_1, \dots, \gamma_k \,|\, (\gamma_i\gamma_j)^{m_{ij}} = 1 \,\forall\, i,j \rangle$$

where $m_{i,i} = 1$ and $m_{i,j} \in \{2, \infty\}$ for all $i \neq j$.

Classical theory.

Gram matrix $B = (-\cos\frac{\pi}{m_{i,j}})_{1 \le i,j \le k}$.

$\to B_t$: replace -1 by $-t$ in B

$\rightsquigarrow \langle \cdot, \cdot \rangle_t$ symmetric bilinear form on $\mathbb{R}^k$ nondegenerate, of signature $(p, q+1)$ for all $t \gg 1$

Canonical representation (Tits):

$$\begin{array}{rcr} \rho_t : \Gamma \to \mathrm{Aut}(\langle \cdot, \cdot \rangle_t) & \xrightarrow{\sim} & O(p, q+1) \\ \gamma_i \mapsto \text{orthog. refl. } / e_i & \mapsto & \text{orthog. refl. } / x_i(t) \end{array}$$

where $\mathrm{Aut}(\langle \cdot, \cdot \rangle) \subset \mathrm{GL}_k(\mathbb{R})$.

Tits–Vinberg: ρ_t is injective and discrete and preserves a properly convex domain

$$\Omega_t = \mathrm{Int}(\rho_t(\Gamma) \cdot P_t) \subset \mathbb{P}(\mathbb{R}^k)$$

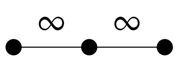
where $P_t = \{[x] \in \mathbb{P}(\mathbb{R}^k) \mid \langle x, e_i \rangle_t \leq 0 \, \forall \, i\}$.

Lemma. $\forall \, t' > t \gg 1$,

- *ρ_t is uniformly contracting w.r.t. $\rho_{t'}$ in spacelike directions,*
- *$u_t := -\frac{d}{dt'}|_{t'=t}$ is uniformly contracting in spacelike directions,*

where $u_t \in T_{\rho_t} \mathrm{Hom}(\Gamma, G) \hookrightarrow \{\rho_t$-cocycles $\Gamma \to \mathfrak{g}\}$.

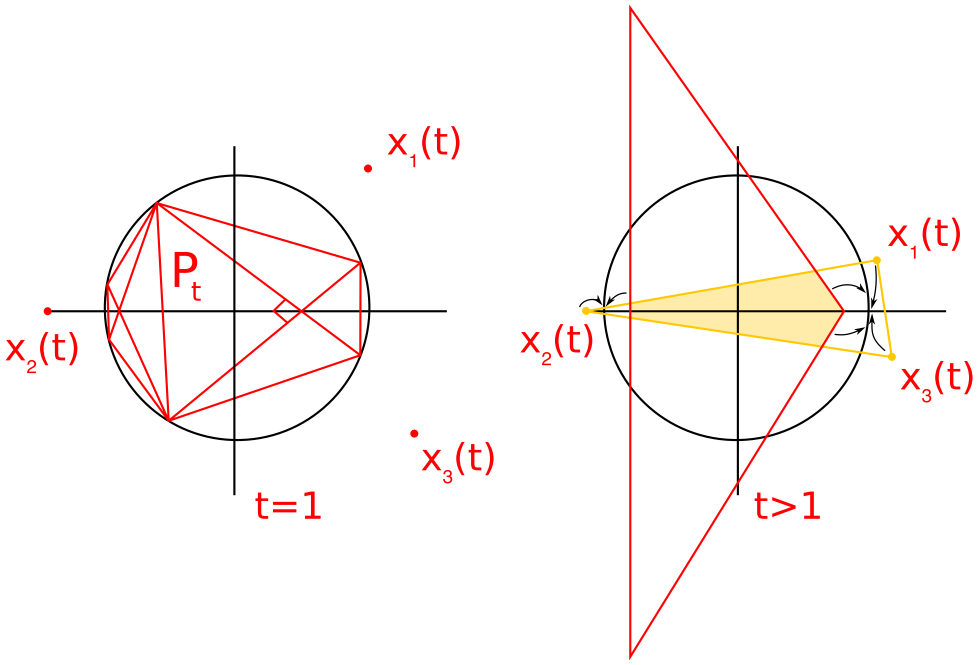
Theorem 2 $\implies$ proper actions on G and $\mathfrak{g}$.

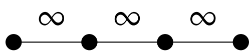
Example 1. 

$B_t = \begin{pmatrix} 1 & -t & \\ -t & 1 & -t \\ & -t & 1 \end{pmatrix}$

$\to$ eigenvalues $1 \pm \sqrt{2}t, 1$

$\implies$ signature $(2,1) \implies$ acts on $\mathbb{H}^2$.





signature (2,2), action on $\mathbb{H}^{2,1}$

$\rightarrow$ proper on $O(2,2)$ and $\mathfrak{o}(2,2)$.