Jonathan Mattingly Lecture 3

Let μ, ν be ergodic inv measures

TFAE

1) $\mu = \nu$

2) $\exists$ a coupling Γ, $\Gamma\left((x_n, y_n)_{n=0}^{\infty} : |x_n - y_n| \xrightarrow{n\to\infty} 0 \right) = 1$

measure on $X^{\mathbb{N}} \times X^{\mathbb{N}}$ so that

$\Gamma \pi_1^{-1} = \mu P_{\mathbb{N}}$, $\Gamma \pi_2^{-1} = \nu P_{\mathbb{N}}$

$$\pi_i : (x, y) \mapsto \begin{cases} x & i=1 \\ y & i=2 \end{cases}$$

3) $\exists \Gamma$ on $X^{\mathbb{N}} \times X^{\mathbb{N}}$, $\Gamma\left((x_n, y_n)_{n=1}^{\infty} : |x_n - y_n| \to 0 \right) > 0$

$\Gamma \pi_1^{-1} \gg \mu P_{\mathbb{N}}$

$\Gamma \pi_2^{-1} \gg \nu P_{\mathbb{N}}$

correction from last time

$(**)$ $\sup_{x,y \in X} \| P(x, \cdot) - P(y, \cdot) \| \le 1 - \alpha$

$(*)$ $P(x, \cdot) \ge \alpha \nu(\cdot)$ $\forall x$

$P(x, A) = \int_A \rho(t, x, y)\, dy$

$x \mapsto P(x, \cdot)$ cts in TV

For all open A, $P(x,A) > 0$ $\forall x$

P is Strong Feller if when ϕ is bdd, $P\phi$ is cts

if P_t is Strong Feller and P_s is strongly top. irreducible

then $P(x, \cdot) \sim P(y, \cdot)$ $\forall x, y$

If $P(x,\cdot)$ is cts at x in TV topology then if μ and ν are two ergodic inv. measures with $x \in \mathrm{supp}(\mu) \cap \mathrm{supp}(\nu) \Rightarrow \mu = \nu$. ie same if overlapping support.

If P is strong Feller then P^2 is cts in TV.
$P_t = P_{t/2} P_{t/2}$.

Suppose we wish to prove cty

$$P\varphi(x) - P\varphi(y) = \int \nabla_x P\varphi(\gamma_s)\dot{\gamma}_s \, ds \leq C|\varphi|_\infty |x-y|$$

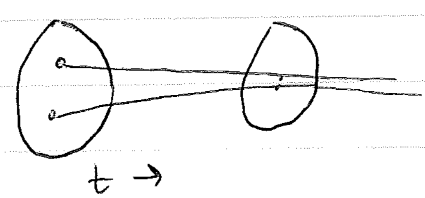
Suppose φ is smooth test func. $\gamma_s : x \to y$ $\quad |\nabla_x P\varphi|_\infty \leq C|\varphi|_\infty$

What if we don't have something like this?

Asymptotic Strong Feller at x

$$(***) \quad |\nabla_x P\varphi| \leq C(x)|\varphi|_\infty + \alpha|\nabla_x \varphi| \qquad \forall \alpha \in (0,1)$$

$\exists \rho$ (positive) st for any ergodic inv measure
If $(***)$, then , $\mathrm{dist}(\mathrm{supp}(\mu), \mathrm{supp}(\nu)) \geq \rho$ if $\mu \neq \nu$.



if we have this estimate we have nearby pts converging in time.

$$\nabla_u \Phi(u)[h] = \lim_{\varepsilon\to 0} \frac{\Phi(u+\varepsilon h) - \Phi(u)}{\varepsilon}$$

$du_t = F(u_t) + \sum_{k=1}^{m} g_k \, dw_t^{(k)}$

$u_t(u_0, W)$

want to peturb W in dir. H.

$$\mathcal{D}_W u_t(u_0, W)[H] = \lim_{\varepsilon\to 0} \frac{u_t(u_0, W+\varepsilon H) - u_t(u_0, W)}{\varepsilon}$$

$H \in H^1(0,t;X) \quad H(0)=0$

Malliarin Derivative

$$\nabla_u u_t(u_0, W)[\xi] = J_{0,t}\,\xi$$

$$\frac{\partial}{\partial t}(J_{0,t}\,\xi) = \nabla_u F(u_t)[J_{0,t}\,\xi], \quad P_t \phi(u_0) = \mathbb{E}_{u_0} \phi(u_t)$$

Let's prove an estimate:

Suppose we can find H_ξ so that $\nabla_u u_t(u_0,W)[\xi] = \mathcal{D}_W u_t(u_0,W)[H]$

$$\nabla_u P_t \Phi(u_0)[\xi] = \mathbb{E}_{u_0} (\nabla_u \phi)(u_t)[J_{0,t}\,\xi] =$$

$\xi, u \in X$

$$= \cancel{\mathbb{E}_{u_0}[\mathcal{D}_W u_t(u_0, W)[H]]}$$

$$= \mathbb{E}_{u_0}\left[\nabla_u \Phi(u_t(u_0,W)) \mathcal{D}_W \phi(u_t)[H]\right]$$

$$= \mathbb{E}_{u_0}\left[\mathcal{D}_W\left(\phi(u_t(u_0,W))\right)[H]\right]$$

$$= \mathbb{E}\left[\phi(u_t(u_0,W)) \int_0^t \dot{H}\, dW_s\right]$$

$$\leq |\phi|_\infty \, \mathbb{E}\left[\int_0^t |\dot{H}_s|^2 ds\right]$$

What if we can find H st

$$J_{0,t}\xi - \mathcal{D}u_t(u_0, W)[H] \xrightarrow{t\to\infty} 0$$