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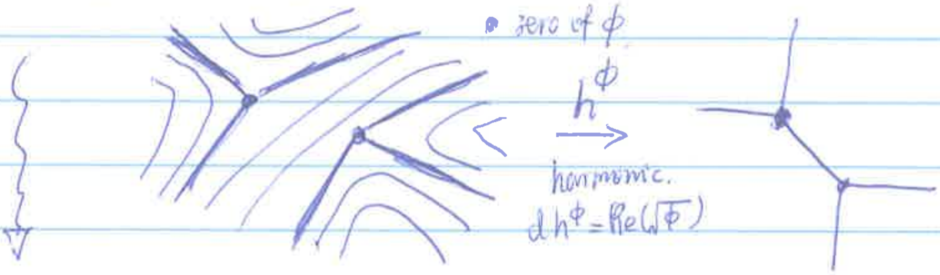
joint w/ Katsarkov, Noll, Simpson

arxiv: 1311.7101

1503.00989

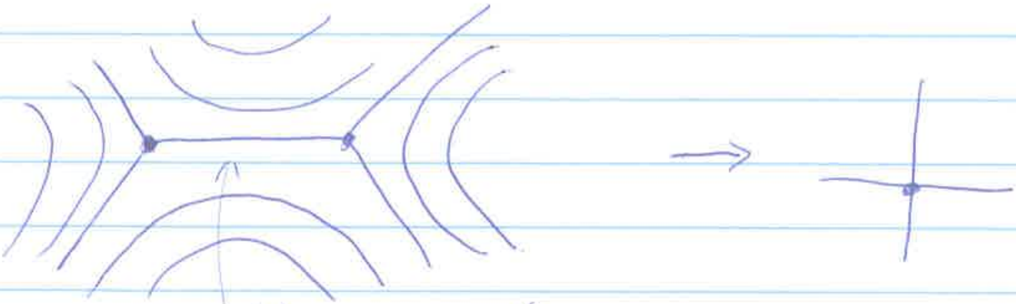
ϕ: quadratic differential on universal cover a Riemann surface X.



leaf space.
tree.

measured foliation. $\left(\mathrm{Re}\sqrt{\phi} = 0,\ \pm\int \mathrm{Re}\sqrt{\phi}\right)$

singular leaves $= h^{\phi^{-1}}$(Singular of tree).



saddle connection / trajectory
Finite length leaf

<u>Goal</u>: Generalise the various features of this picture to "higher rank"

Motivation: 1) Categorical algebraic geometry.

nc-space $\mathcal{C}$ $\quad$ triangulated dg-category $\xrightarrow{\text{Bridgeland}}$ $\mathrm{Stab}(\mathcal{C})$ cplx mfld.

$\mathcal{P}_\theta \subseteq \mathcal{C}$ $\theta \in S^1$ subcats of semi stable obj of phase θ

$K_0(\mathcal{C}) \xrightarrow{Z} \mathbb{C}$

Kontsevich–Soibelman $(\mathcal{C}, \sigma \in \mathrm{Stab}(\mathcal{C}), [\alpha] \in K_0(\mathcal{C}))$ $\rightsquigarrow$ $DT(\mathcal{C}, \sigma, \alpha)$ counts semi-stable objects whose class is $[\alpha]$

+ "wall crossing formula" controlling the jumps of DT along codim 1 walls in $\mathrm{Stab}(\mathcal{C})$

Bridgeland–Smith, Haiden–Katzarkov–Kontsevich.
Spaces of quadratic differentials can be identified with stability conditions on certain ("Fukaya") category.

Goal: Realize the $SL_r\mathbb{C}$ Hitchin base as sitting in the space of stability condition.

2) Compactifying $\mathcal{M}_{Hitchin} \simeq \mathcal{M}_{Betti} \simeq \mathcal{M}_{Higgs} \simeq \mathcal{M}_{DR}$

$$\begin{array}{ccccccc}
\mathcal{M}_{DR} & \hookrightarrow & \mathcal{M}_{Hodge} & \hookleftarrow & \mathcal{M}_{Higgs} & \xrightarrow{\text{char}} & \mathrm{Hit}_r \\
\downarrow & & \downarrow & & \downarrow & & \mathrel{\rotatebox{90}{$\subseteq$}} \\
\{1\} & \to & \mathbb{A}^1 & \leftarrow & \{0\} & & \bigoplus_{k=2}^{r} H^0(X, K_X^k)
\end{array}$$

$\mathcal{M}_{Hodge} = \{(\mathcal{E}, \nabla) \mid$ $\mathcal{E}$-alg v.b $\nabla(fs) = \lambda\, df\, s + f\nabla s\}$

Rk: $\lambda = 0$ $\Phi \in \mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)$

$$\overline{\mathcal{M}}_{DR} - \mathcal{M}_{DR} = D \simeq \left(\mathcal{M}_{Higgs} - \mathrm{char}^{-1}(0)\right) / \mathbb{C}^*$$

$\uparrow S^1$-bundle

blow up $\rightsquigarrow$ $\widetilde{\mathcal{M}}_{DR} - \mathcal{M}_{DR} = \widetilde{D} \simeq \left(\mathcal{M}_{Higgs} - \mathrm{Char}^{-1}(0)\right) / \mathbb{R}_+$

$\mathcal{M}_{Betti} = \{\pi_1(X) \to SL_r(\mathbb{C})\} // \mathrm{conj}$ — Morgan–Shalen

$\overline{\mathcal{M}}_{Betti} - \mathcal{M}_{Betti}$ = action of $\pi_1(X)$ on $\mathbb{R}$-building. — Parreau

<u>Goal</u>: If γ is a curve approaching a point $[\Phi] \in D$ in $\overline{\mathcal{M}}_{DR}$
which point does this correspond to in $\overline{\mathcal{M}}_{Betti} - \mathcal{M}_{Betti}$

Fix $\mathcal{E}$, ∇_0, Φ.

$\rightsquigarrow$ $\nabla_\hbar = \nabla_0 + \Phi/\hbar$

$\rightsquigarrow$ ~~$\pi_1(X) \to$ Hom~~

$\pi_1(X, x) \longrightarrow \mathrm{Hom}(\mathcal{E}_x, \mathcal{E}_x)$

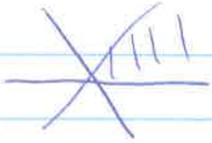
$\gamma \longmapsto \mathrm{Transport}_\gamma(\hbar) \sim e^{\nu_{BKW}/\hbar}$.

<u>Fix</u> any hermitien metric h on $\mathcal{E}$.

$$\nu_{WKB} = \limsup_{\hbar \to 0} \hbar \log \| \mathrm{Trans}_\gamma(\hbar) \|$$

WKB-exponent

WKB-dilatation spectrum



$\vec{\nu}_{WKB}(\gamma) = (\alpha_1, \ldots, \alpha_r)$

Characterized $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r$

$$\sum_{i=1}^{k} \alpha_k = \lim_{\hbar \to 0} \hbar \log \| \wedge^k \mathrm{Trans}(\hbar) \|$$

<u>Goal</u>: Describe $\vec{\nu}_{WKB}(\gamma)$ in terms of $\mathrm{char}(\Phi)$ - spectral cover

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook] \arrow[dr] & T_X^\vee \arrow[d] \\ & X \end{tikzcd}$$

Transporting the metric h to Fixed fiber $\hat{\mathcal{E}}_p \simeq \mathbb{C}^r$ using $\nabla_\hbar$.

$h_{\hbar U}: \tilde{X}^\omega \xrightarrow[\hbar \to 0]{\lim h_\hbar} \underbrace{\lim_{\hbar\to 0}{}^{\omega} (SL_r\mathbb{C}/SU(r), \hbar d)}$

Thm: h_ω is ϕ-harmonic. $\mathrm{Cone}_\omega \leftarrow \mathbb{R}$-building $\begin{pmatrix}\text{Kleiner}\\ \text{Leeb}\end{pmatrix}$ ω ultrafilter.

~~Tautology~~ Tautology: $\nu^\omega_{WKB}(\gamma) \doteq d_{h_\omega}(h_\omega(\gamma(0)), h_\omega(\gamma(1)))$

<u>Thm</u> $\forall \gamma \in \pi_1(X)$ $\exists\, \omega$ s.t. $\nu^\omega_{WKB} = \nu_{WKB}$

<u>Def</u>: A $\pi_1(X)$-equivariant map $\tilde{X} \to B$ Building is a harmonic ϕ-map where $\phi \in \bigoplus_{k=1}^{r} H^0(X, K_X^k)$, if $\forall x \in X$ away from a codim 2 subset

$$\exists \quad \begin{tikzcd} x \in U_x \arrow[r, "h|_{U_x}"] \arrow[d, phantom, "\cap"] & A \arrow[d] \\ \tilde{X} \arrow[r] & B \end{tikzcd} \qquad dh_{U_x} = \{\lambda_1, \dots, \lambda_r\}.$$

<u>Def</u>: $\tilde{X} \xrightarrow{h^\phi} B^\phi$ is a (uni) versal harmonic ϕ-map

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[dr, "h"'] & B^\phi \arrow[d, dashed, "\exists(!)"] \\ & B \end{tikzcd}$$

Berk – Nevins – Roberts (1983)

$$\hbar^3 \frac{d^3}{dx} - 3\hbar \frac{d}{dx} + x = 0 \qquad SL_3(\mathbb{C}) \text{ WKB-probleme.}$$

<u>Conjecture</u>: For generic ϕ (no BPS state) $\in SL_3\mathbb{C}$ – Hitchin base, there exists a universal pre-building.