Diophantine approximation on group varieties

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joint work with Alexander Gorodnik and Amos Nevo

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- $S_N(x) = \sum_{n=1}^{N} h_n(x)$ and $E_N = \sum_{n=1}^{N} \int_X h_n d\mu$

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- Dani: x is badly approximable if and only if the g_t orbit of $u_x\mathbb{Z}^{n+1}$ is bounded

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$$\text{shortest vector}(g_t u_x \mathbb{Z}^{n+1}) \leq r(t)$$

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$$\sum_{t=0}^{\infty} \mathrm{vol}(X_{n+1}(t))$$

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- Ghosh-Gorodnik-Nevo (2015): Analogue of Schmidt's theorem for group varieties

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- Fishman-Kleinbock-Merrill-Simmons: rational approximation on quadratic surfaces

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- θ is the "spectral gap", given by the integrability of matrix coefficients

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- This theorem holds more generally for S-integer and rational points

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- Provided G is a product of split rank 1 groups and Γ is cocompact