Title: Gross-Zagier Formula: Why Is It Right?

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References

- MSRI Publ 49
- Ann Math Studies 184

$E/\mathbb{Q}$ Elliptic Curve

$\Gamma \hookrightarrow SL_2(\mathbb{Z})$ $\quad X_\Gamma = " \ {}_{\Gamma}\backslash \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$

$\hookrightarrow SL_2(\mathbb{Q})$ $\quad \mathrm{Hom}_0(X_\Gamma, E) \neq 0$

$X_\Gamma / \mathbb{Q}^\Gamma / \mathbb{Q}$

$$X := \varprojlim_{\Gamma} X_{\Gamma,\mathbb{Q}}(\mathbb{C}) = \varprojlim_{U \subset GL_2(\hat{\mathbb{Q}})} GL_2(\mathbb{Q})\backslash \mathcal{H}^{\pm} \times GL_2(\hat{\mathbb{Q}})/U$$

$\mathbb{Q}^\Gamma \hookrightarrow \mathbb{C}$

$$= {}_{GL_2(\mathbb{Q})}\backslash \mathcal{H} \times GL_2(\hat{\mathbb{Q}})$$

$$= \mathcal{H} \times \hat{\mathbb{Z}}$$

$$\pi_E := \mathrm{Hom}_0(X, E) \otimes \mathbb{Q} = \varinjlim \mathrm{Hom}_0(X_\Gamma, E) \otimes \mathbb{Q}$$

π_E has an action by $GL_2(\hat{\mathbb{Q}})$!

$$\hat{\mathbb{Q}} = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \quad \hat{\mathbb{Z}} = \prod \mathbb{Z}_p = \varprojlim \mathbb{Z}/N\mathbb{Z}$$

π_E is an irreducible admissible rep of $GL_2(\hat{\mathbb{Q}})$

$\pi_E = \bigotimes_{p<\infty} \pi_p$ $\qquad$ π_p irred rep of $GL_2(\mathbb{Q}_p)$

$f \in \pi \qquad g \in GL_2(\hat{\mathbb{Q}})$

$(gf)(x) = f(xg)$

$K = \mathbb{Q}(\sqrt{d}),\ d<0.$

Fix one embedding $K \hookrightarrow M_{2\times 2}(\hat{\mathbb{Q}})$

$K^\times \hookrightarrow \hat{K}^\times \hookrightarrow GL_2(\hat{\mathbb{Q}})$

$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ $\circlearrowleft$ $\mathbb{Q}\backslash X^{K^\times}$ = subscheme of points fixed by $K^\times$

$\mathrm{Norm}_{K^\times}(GL_2(\hat{\mathbb{Q}})) = \hat{K}^\times \amalg \hat{K}^\times j$

$jx = \bar{x} j \ \xleftarrow{\sigma}$

<u>CM-theory</u>: $\mathrm{Gal}(K^{ab}/\mathbb{Q}) \xrightarrow{\sim} \langle \hat{K}^\times, j\rangle / K^\times$

<u>Ex</u>: $K = \mathbb{Q}(\sqrt{d}) \hookrightarrow M_2(\mathbb{Q})$

$a + b\sqrt{d} \longmapsto \begin{pmatrix} a & b \\ bd & a \end{pmatrix}$

$K^\times \circlearrowright \mathcal{H}$ has a fixed point

$\frac{az+b}{bdz+a} = z, \qquad z = -\frac{1}{\sqrt{d}} \in \mathcal{H}$

$$\mathcal{H} \longrightarrow X_\Gamma$$
$$z \longmapsto p_\Gamma$$
$$p = \varprojlim p_\Gamma \in X(K^{ab})$$

$$\begin{array}{ccc} K^\times \backslash \hat{K}^\times & \hookrightarrow & X \\ \| & t \longmapsto p^{\sigma(t)} & \\ \mathrm{Spec}(K^{ab}) & & \end{array}$$

$$f \in \pi, \quad \chi : \mathrm{Gal}(K^{ab}/K) \longrightarrow \mathbb{C}^\times$$

$$P(f, \chi) = \int_{\mathrm{Gal}(K^{ab}/K)} f(p^\sigma)\chi(\sigma)\, d\sigma$$

$$= \int_{K^\times \backslash \hat{K}^\times} f(t)\, \chi(t)\, dt$$

<u>Rmk</u>: π should be considered as space of automorphic forms on $GL_2(\hat{\mathbb{Q}})$ with values in E.

<u>Observation</u> I

$$f \longmapsto P(f, \chi)$$

is a linear functional

$$\mathrm{Hom}_{\hat{K}^\times}\big(\underset{GL_2(\hat{\mathbb{Q}})}{\underset{\circlearrowleft}{\pi}} \otimes \underset{\hat{K}^\times}{\underset{\circlearrowleft}{\chi}}, \mathbb{C}\big) \otimes E(K^{ab})$$

$$\langle \ \ \rangle_{NT} : E(K^{ab}) \times E(K^{ab}) \longrightarrow \mathbb{R}$$

$$\langle P(f_1, \chi), P(f_2, \chi^{-1}) \rangle_{NT} \longleftarrow\!\shortmid (f_1, f_2)$$

$e_{\mathbb{C}}^{NT}$

bilinear $\mathbb{C}$-lin on $E(K^{ab}) \otimes \mathbb{C}$

This gives an element

$$\beta \in \mathrm{Hom}_{\hat{K}^\times}(\pi \otimes \chi, \mathbb{C}) \otimes \mathrm{Hom}_{\hat{K}^\times}(\pi \otimes \chi^{-1}, \mathbb{C})$$

<u>Observation</u> II (Saito-Tunnell-Waldspurger)

① $\dim \mathrm{Hom}_{\hat{K}^\times}(\pi \otimes \chi, \mathbb{C}) \leq 1$

$$\| $$

$$\otimes \mathrm{Hom}_{K_p^\times}(\pi_p \otimes \chi_p, \mathbb{C})$$

ε-factor of $L(E, \chi, s)$

$$\dim = 1 \iff \varepsilon(\pi_p, \chi_p) = 1$$

$\dim \mathrm{Hom}_{\hat{K}^\times}(\pi \otimes \chi, \mathbb{C}) = 1$ S.T. condition

$\iff$ $\varepsilon(\pi_p, \chi_p) = 1$ $\forall p < \infty$

and $\chi|_{\hat{\mathbb{Q}}^\times} = 1$

② There is an explicit generator

$$\alpha_p \in \mathrm{Hom}_{K_p^\times}(\pi_p \otimes \chi_p, \mathbb{C}) \otimes \mathrm{Hom}_{K_p^\times}(\pi_p \otimes \chi^{-1}, \mathbb{C})$$

$$\alpha_p(f_1, f_2) = \int_{\mathbb{Q}_p^\times \backslash K_p^\times} (\pi_p(t) f_1, f_2) \chi(t) dt$$

$$\pi \otimes \pi \longrightarrow \mathbb{C}$$

$$\langle f, f \rangle = \frac{\deg f}{\mathrm{vol}(X^\Gamma)} \qquad f_\Gamma : X_\Gamma \longrightarrow E$$

Normalize $\alpha_p^{\#} = c_p \alpha_p$ | $\alpha_p^{\#}(f_1, f_2) = 1$
f_i: spherical base

such that $\prod \alpha_p^{\#}$

define a generator of $\mathrm{Hom}_{\hat{K}^\times}(\pi \otimes \chi, \mathbb{C}) \otimes \mathrm{Hom}_{\hat{K}^\times}(\pi \otimes \chi^{-1}, \mathbb{C})$

Combine Observations I and II

If S.T. condition holds

then there is a constant

$c \in \mathbb{C}$ such that $\beta = c \cdot \prod \alpha_p^{\#}$

Equivalently $\exists c \in \mathbb{C}$ s.t. for any $f_1, f_2 \in \pi$

$$\langle P(f_1, \chi_1), P(f_2, \chi_2) \rangle_{NT}$$
$$= c \cdot \alpha^{\#}(f_1, f_2)$$

Gross-Zagier formula

is about adelic decomposition

of a bilinear form on $\pi \otimes \pi$

Gross-Zagier formula (Revisited)

$P(f, \chi) \neq 0$ iff

1) $\mathrm{Hom}(\pi \otimes \chi, \mathbb{C}) \neq 0$ ($\Leftrightarrow \varepsilon(\pi_p, \chi_p) = 1$ $\forall p < \infty$)

2) $L'(E, \chi, 1) \neq 0$

Moreover for any $f_1, f_2 \in \pi$

$$\langle P(f_1,\chi), P(f_2,\chi)\rangle_{NT} = L'(\pi,\chi,1)\,\alpha(f_1,f_2)$$

$$\parallel$$

$$\beta(f_1,f_2)$$

<u>Rmk</u>: Replace $X = {}_{GL_2(\mathbb{Q})}\backslash \mathcal{H}^{\pm} \times GL_2(\mathbb{A}_f)$

$$= {}_{GL_2(\mathbb{Q})}\backslash GL_2(\mathbb{A}) / K_\infty$$

by ${}_{GL_2(\mathbb{Q})}\backslash GL_2(\mathbb{A})$

and E by $\mathbb{C}$.

$$\underset{\text{irreducible}}{\circlearrowright}\ \pi \hookrightarrow \mathrm{Hom}({}_{GL_2(\mathbb{Q})}\backslash GL_2(\mathbb{A}), \mathbb{C})$$

$P(f,\chi) \in \mathbb{C}$ Waldspurger formula

$$P(f_1,\chi)\,P(f_2,\chi^{-1}) = L(\pi,\chi,1)\,\alpha(f_1,f_2)$$

<u>Rmk</u>: There are other extensions

1) Shimura curve if $L(E,\chi,s)$ has odd functional equation then $\exists!$ quaternion algebra B (Shimura) s.t.

$$\mathrm{Hom}(\pi^{JL} \otimes \chi, \mathbb{C}) \neq 0$$

$P(f, \chi) \neq 0 \Leftarrow\Rightarrow L'(E, \chi, 1) \neq 0$

(YZZ)

2) p-adic GZ (Perrin-Riou, Kobayashi, Disegni)

$$\langle \quad \rangle_{NT} \Longleftrightarrow \langle \quad \rangle_{p\text{-adic } NT}$$

3) p-adic Waldspurger formula

$$\underset{\mathbb{C}}{\langle \quad \rangle_{NT}} \Longleftrightarrow \log P(f_1, \chi) \cdot \log(f_2, \chi^{-1})$$

$$X(\mathbb{C}_p) \xrightarrow{f} E(\mathbb{C}_p) \xrightarrow{\log} \mathbb{C}_p$$

(Bertolini-Darmon-Prasanna, Lin, Wei, Zhang, PZ)