Maximal stream, minimal cutset and maximal flow in d-dimensional first passage percolation

Marie Théret

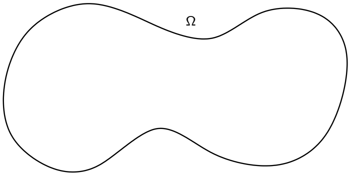
LPMA, Université Paris Diderot (Paris VII)

joint work with Raphaël Cerf (IUF - Université Paris Sud)

A model for porous media

- $\Omega \subset \mathbb{R}^d$ open bounded connected ($d \geq 2$) $\leftrightsquigarrow$ piece of **rock**
- $\Gamma^1, \Gamma^2 \subset \partial\Omega$ open $\leftrightsquigarrow$ where the water can **enter** / **come out**
- Graph $(\mathbb{V}_n, \mathbb{E}_n) = (\mathbb{Z}^d/n, \mathbb{E}^d/n) \cap \Omega$ $\leftrightsquigarrow$ **tubes**
- Random variables $(t(e))_{e \in \mathbb{E}_n}$ i.i.d. ≥ 0 $\leftrightsquigarrow$ **capacities** :

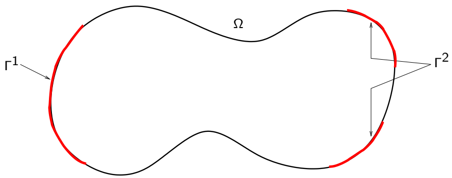
$t(e)$ is the maximal amount of water than can cross e per second.



A model for porous media

- $\Omega \subset \mathbb{R}^d$ open bounded connected ($d \geq 2$) $\leftrightsquigarrow$ piece of **rock**
- $\Gamma^1, \Gamma^2 \subset \partial\Omega$ open $\leftrightsquigarrow$ where the water can **enter** / **come out**
- Graph $(\mathbb{V}_n, \mathbb{E}_n) = (\mathbb{Z}^d/n, \mathbb{E}^d/n) \cap \Omega \leftrightsquigarrow$ **tubes**
- Random variables $(t(e))_{e \in \mathbb{E}_n}$ i.i.d. $\geq 0 \leftrightsquigarrow$ **capacities** :

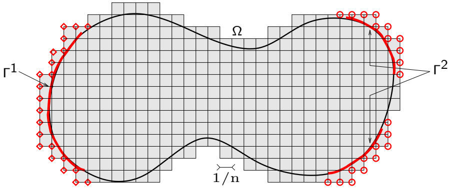
$t(e)$ is the maximal amount of water than can cross e per second.



A model for porous media

- $\Omega \subset \mathbb{R}^d$ open bounded connected $(d \geq 2)$ $\leftrightsquigarrow$ piece of **rock**
- $\Gamma^1, \Gamma^2 \subset \partial\Omega$ open $\leftrightsquigarrow$ where the water can **enter** / **come out**
- Graph $(\mathbb{V}_n, \mathbb{E}_n) = (\mathbb{Z}^d/n, \mathbb{E}^d/n) \cap \Omega$ $\leftrightsquigarrow$ **tubes**
- Random variables $(t(e))_{e \in \mathbb{E}_n}$ i.i.d. ≥ 0 $\leftrightsquigarrow$ **capacities** :

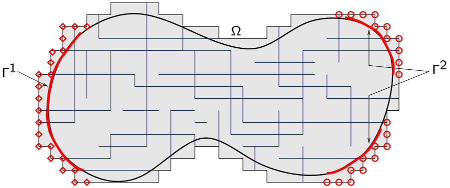
$t(e)$ is the maximal amount of water than can cross e per second.



A model for porous media

- $\Omega \subset \mathbb{R}^d$ open bounded connected ($d \geq 2$) $\leftrightsquigarrow$ piece of **rock**
- $\Gamma^1, \Gamma^2 \subset \partial\Omega$ open $\leftrightsquigarrow$ where the water can **enter** / **come out**
- Graph $(\mathbb{V}_n, \mathbb{E}_n) = (\mathbb{Z}^d/n, \mathbb{E}^d/n) \cap \Omega \leftrightsquigarrow$ **tubes**
- Random variables $(t(e))_{e \in \mathbb{E}_n}$ i.i.d. $\geq 0 \leftrightsquigarrow$ **capacities** :

$t(e)$ is the maximal amount of water than can cross e per second.

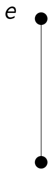


$t(e) = 0$ ou 1

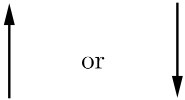
Streams

Circulation of water : $e \in \mathbb{E}_n \mapsto \vec{f}_n(e)$ such that

- $\|\vec{f}_n(e)\|$ = amount of water that cross e per second,
- $\frac{\vec{f}_n(e)}{\|\vec{f}_n(e)\|}$ = direction in which the water circulates.



$\frac{\vec{f}_n(e)}{\|\vec{f}_n(e)\|}$:



Streams

Circulation of water : $e \in \mathbb{E}_n \mapsto \vec{f}_n(e)$ such that

- $\|\vec{f}_n(e)\|$ = amount of water that cross e per second,
- $\frac{\vec{f}_n(e)}{\|\vec{f}_n(e)\|}$ = direction in which the water circulates.

Constraints :

- node law, at each point in $\mathbb{V}_n \setminus (\Gamma^1 \cup \Gamma^2)$,
- capacity constraint *(random)* : $\forall e \in \mathbb{E}_n$, $\|\vec{f}_n(e)\| \leq t(e)$.

Stream : Borel vector measure defined by

$$\vec{\mu}_n = \sum_{e \in \mathbb{E}_n} \vec{f}_n(e)\, \delta_{\mathrm{center}(e)} \,.$$

Maximal flow

Flow : $\mathrm{flow}_n^{\mathrm{disc}}(\vec{\mu}_n)$ is the amount of water that enters in Ω through Γ^1 per second according to $\vec{\mu}_n$.

Maximal flow :

$$\phi_n = \sup\{\mathrm{flow}_n^{\mathrm{disc}}(\vec{\mu}_n)\}.$$

Cutsets :

- $E_n \subset \mathbb{E}_n$ is a **cutset** if $\Gamma^1 \not\leftrightarrow \Gamma^2$ in $\mathbb{E}_n \setminus E_n$,
- $\mathrm{capacity}_n^{\mathrm{disc}}(E_n) = \sum_{e \in E_n} t(e)$,
- Max-flow min-cut Theorem (Ford and Fulkerson, '56) :

$$\phi_n = \min\left\{ \mathrm{capacity}_n^{\mathrm{disc}}(E_n) \,\middle|\, E_n \text{ is a cutset} \right\}.$$

Maximal flow

Flow : $\mathrm{flow}_n^{\mathrm{disc}}(\vec{\mu}_n)$ is the amount of water that enters in Ω through Γ^1 per second according to $\vec{\mu}_n$.

Maximal flow :

$$\phi_n = \sup\{\mathrm{flow}_n^{\mathrm{disc}}(\vec{\mu}_n)\}.$$

Cutsets :

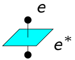
- $E_n \subset \mathbb{E}_n$ is a **cutset** if $\Gamma^1 \not\leftrightarrow \Gamma^2$ in $\mathbb{E}_n \setminus E_n$,
- $\mathrm{capacity}_n^{\mathrm{disc}}(E_n) = \sum_{e \in E_n} t(e)$,
- Max-flow min-cut Theorem (Ford and Fulkerson, '56) :

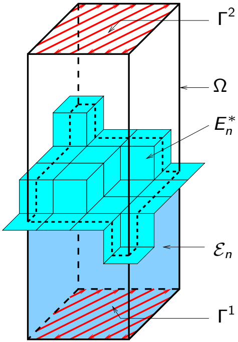
$$\phi_n = \min\left\{\mathrm{capacity}_n^{\mathrm{disc}}(E_n) \,|\, E_n \text{ is a cutset}\right\}.$$

Cutsets

Representation of a cutset :

- "dual" of an edge e
  $=$ small "plaquette" e^*,
- "dual" of a cutset E_n
  $=$ "surface" E_n^*
  $=$ boundary of a set $\mathcal{E}_n$:
  $\mathcal{E}_n \subset \Omega$, $\Gamma^1 \subset \partial\mathcal{E}_n$.





Goal

Main characters :

- maximal flow ϕ_n *(random real number)*,
- maximal stream $\vec{\mu}_n^{\max}$ *(random vector measure)*,
  i.e., stream of maximal flow, and such that no water comes out of Ω through Γ^1,
- minimal cutset $\mathcal{E}_n^{\min}$ *(random subset of Ω)*,
  i.e., cutset of minimal capacity, and of minimal number of edges.

Question : Behaviors of ϕ_n, $\vec{\mu}_n^{\max}$ and $\mathcal{E}_n^{\min}$ when $n \to \infty$?

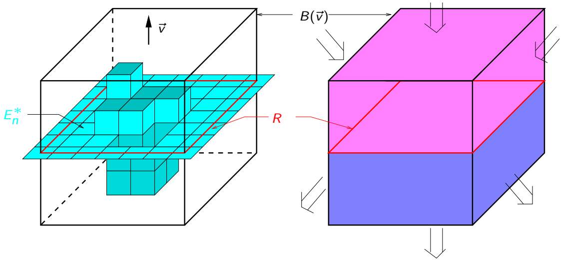
Continuous capacity $\nu(\vec{v})$

Definitions :

$B(\vec{v})$ unit cube oriented towards $\vec{v} \in \mathbb{S}^{d-1}$,

$$\tau_n(B(\vec{v})) = \min \left\{ \text{capacity}_n^{\text{disc}}(E_n) \,\middle|\, \begin{array}{c} E_n^* \text{ surface of plaquettes} \\ \text{in } B(\vec{v}) \text{ of boundary } R \end{array} \right\}.$$

$\tau_n(B(\vec{v})) =$ maximal flow from pink to blue.



Continuous capacity $\nu(\vec{v})$

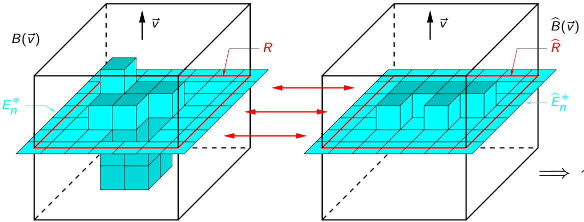
Definitions :

$B(\vec{v})$ unit cube oriented towards $\vec{v} \in \mathbb{S}^{d-1}$,

$$\tau_n(B(\vec{v})) = \min \left\{ \text{capacity}_n^{\text{disc}}(E_n) \,\middle|\, \begin{array}{c} E_n^* \text{ surface of plaquettes} \\ \text{in } B(\vec{v}) \text{ of boundary } R \end{array} \right\}.$$

Subadditivity :

$$\frac{\tau_n(B(\vec{v}))}{n^{d-1}} \xrightarrow[n\to\infty]{p.s.} \nu(\vec{v})\,.$$



$\Longrightarrow \tau_n(B(\vec{v}) \cup \widehat{B}(\vec{v}))$
$\leq \tau_n(B(\vec{v})) + \tau_n(\widehat{B}(\vec{v}))$

Continuous capacity $\nu(\vec{v})$

Definitions :

$B(\vec{v})$ unit cube oriented towards $\vec{v} \in \mathbb{S}^{d-1}$,

$$\tau_n(B(\vec{v})) = \min \left\{ \text{capacity}_{\text{n}}^{\text{disc}}(E_n) \;\middle|\; \begin{array}{c} E_n^* \text{ surface of plaquettes} \\ \text{in } B(\vec{v}) \text{ of boundary } R \end{array} \right\}.$$

Subadditivity :

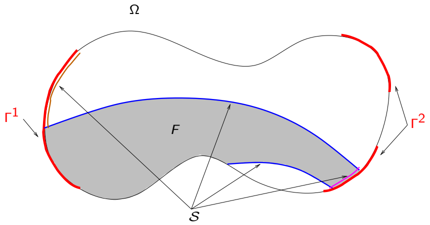
$$\frac{\tau_n(B(\vec{v}))}{n^{d-1}} \xrightarrow[n\to\infty]{p.s.} \nu(\vec{v}) \,.$$

Interpretation : $\nu(\vec{v})$ is the average asymptotic capacity of a continuous unit surface normal to $\vec{v}$.

Continuous cutset

Cutstet : $F \subset \Omega$ of finite perimeter ($\mathbb{1}_F \in BV(\Omega)$)
$\longrightarrow \mathcal{S} = (\partial F \cap \Omega) \cup (\partial F \cap \Gamma^2) \cup (\partial(\Omega \setminus F) \cap \Gamma^1)$.

Capacity : $\text{capacity}^{\text{cont}}(F) = \int_{\mathcal{S}} \nu(\vec{v}_{\mathcal{S}}(x))\, d\mathcal{H}^{d-1}(x)$.

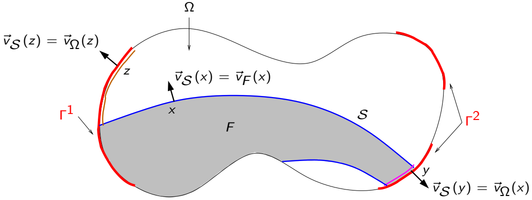


Variational problem :

$$\phi^{\text{cutset}} = \inf\{\text{capacity}^{\text{cont}}(F) \,|\, F \subset \Omega,\, \mathbb{1}_F \in BV(\Omega)\},$$
$$\Sigma^{\text{cutset}} = \{F \subset \Omega \,|\, \mathbb{1}_F \in BV(\Omega),\, \text{capacity}^{\text{cont}}(F) = \phi^{\text{cutset}}\}.$$

Continuous cutset

Cutstet : $F \subset \Omega$ of finite perimeter $(\mathbb{1}_F \in BV(\Omega))$
$$\longrightarrow \mathcal{S} = (\partial F \cap \Omega) \cup (\partial F \cap \Gamma^2) \cup (\partial(\Omega \smallsetminus F) \cap \Gamma^1).$$
Capacity : $\text{capacity}^{\text{cont}}(F) = \int_{\mathcal{S}} \nu(\vec{v}_{\mathcal{S}}(x))\, d\mathcal{H}^{d-1}(x)$.



Variational problem :

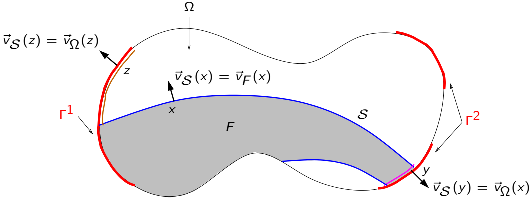
$$\phi^{\text{cutset}} = \inf\{\text{capacity}^{\text{cont}}(F) \,|\, F \subset \Omega,\, \mathbb{1}_F \in BV(\Omega)\},$$
$$\Sigma^{\text{cutset}} = \{F \subset \Omega \,|\, \mathbb{1}_F \in BV(\Omega),\, \text{capacity}^{\text{cont}}(F) = \phi^{\text{cutset}}\}.$$

Continuous cutset

Cutstet : $F \subset \Omega$ of finite perimeter ($\mathbb{1}_F \in BV(\Omega)$)

$$\longrightarrow \mathcal{S} = (\partial F \cap \Omega) \cup (\partial F \cap \Gamma^2) \cup (\partial(\Omega \setminus F) \cap \Gamma^1).$$

Capacity : $\text{capacity}^{\text{cont}}(F) = \int_{\mathcal{S}} \nu(\vec{v}_{\mathcal{S}}(x))\, d\mathcal{H}^{d-1}(x)$.



Variational problem :

$$\phi^{\text{cutset}} = \inf\{\text{capacity}^{\text{cont}}(F) \,|\, F \subset \Omega\,,\, \mathbb{1}_F \in BV(\Omega)\}\,,$$

$$\Sigma^{\text{cutset}} = \{F \subset \Omega \,|\, \mathbb{1}_F \in BV(\Omega)\,,\, \text{capacity}^{\text{cont}}(F) = \phi^{\text{cutset}}\}\,.$$

Continuous stream

Stream : vector field $\vec{\sigma} \in L^\infty(\Omega \to \mathbb{R}^d, \mathcal{L}^d)$ satisfying

- *boundary conditions :* $\vec{\sigma} \cdot \vec{v}_\Omega \leq 0$ $\mathcal{H}^{d-1}$-a.e. on Γ^1 and $\vec{\sigma} \cdot \vec{v}_\Omega = 0$ $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega \setminus (\Gamma^1 \cup \Gamma^2)$,
- *conservation law :* $\operatorname{div} \vec{\sigma} = 0$ $\mathcal{L}^d$-a.e. on Ω,
- *capacity constraint :* $\vec{\sigma} \cdot \vec{v} \leq \nu(\vec{v})$ for all $\vec{v} \in \mathbb{S}^{d-1}$, $\mathcal{L}^d$-a.e.

Flow : $\text{flow}^{\text{cont}}(\vec{\sigma}) = \int_{\Gamma^1} -\vec{\sigma} \cdot \vec{v}_\Omega \, d\mathcal{H}^{d-1}$.

Variational problem :

$$\phi^{\text{stream}} = \sup\{\text{flow}^{\text{cont}}(\vec{\sigma}) \,|\, \vec{\sigma} \text{ admissible stream in } \Omega\},$$
$$\Sigma^{\text{stream}} = \{\vec{\sigma} \,|\, \vec{\sigma} \text{ admissible stream in } \Omega, \ \text{flow}^{\text{cont}}(\vec{\sigma}) = \phi^{\text{stream}}\}.$$

Continuous stream

Stream : vector field $\vec{\sigma} \in L^\infty(\Omega \to \mathbb{R}^d, \mathcal{L}^d)$ satisfying

- *boundary conditions :* $\vec{\sigma} \cdot \vec{v}_\Omega \leq 0$ $\mathcal{H}^{d-1}$-a.e. on Γ^1 and
  $\vec{\sigma} \cdot \vec{v}_\Omega = 0$ $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega \smallsetminus (\Gamma^1 \cup \Gamma^2)$,
- *conservation law :* $\operatorname{div} \vec{\sigma} = 0$ $\mathcal{L}^d$-a.e. on Ω,
- *capacity constraint :* $\vec{\sigma} \cdot \vec{v} \leq \nu(\vec{v})$ for all $\vec{v} \in \mathbb{S}^{d-1}$, $\mathcal{L}^d$-a.e.

Flow : $\text{flow}^{\text{cont}}(\vec{\sigma}) = \int_{\Gamma^1} -\vec{\sigma} \cdot \vec{v}_\Omega \, d\mathcal{H}^{d-1}$.

Variational problem :

$$\phi^{\text{stream}} = \sup\{\text{flow}^{\text{cont}}(\vec{\sigma}) \,|\, \vec{\sigma} \text{ admissible stream in } \Omega\},$$
$$\Sigma^{\text{stream}} = \{\vec{\sigma} \,|\, \vec{\sigma} \text{ admissible stream in } \Omega, \text{flow}^{\text{cont}}(\vec{\sigma}) = \phi^{\text{stream}}\}.$$

Continuous stream

Stream : vector field $\vec{\sigma} \in L^\infty(\Omega \to \mathbb{R}^d, \mathcal{L}^d)$ satisfying

- *boundary conditions :* $\vec{\sigma} \cdot \vec{v}_\Omega \leq 0$ $\mathcal{H}^{d-1}$-a.e. on Γ^1 and
  $\vec{\sigma} \cdot \vec{v}_\Omega = 0$ $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega \setminus (\Gamma^1 \cup \Gamma^2)$,
- *conservation law :* $\operatorname{div} \vec{\sigma} = 0$ $\mathcal{L}^d$-a.e. on Ω,
- *capacity constraint :* $\vec{\sigma} \cdot \vec{v} \leq \nu(\vec{v})$ for all $\vec{v} \in \mathbb{S}^{d-1}$, $\mathcal{L}^d$-a.e.

Flow : $\mathrm{flow}^{\mathrm{cont}}(\vec{\sigma}) = \int_{\Gamma^1} -\vec{\sigma} \cdot \vec{v}_\Omega d\mathcal{H}^{d-1}$.

Variational problem :

$$\phi^{\mathrm{stream}} = \sup\{\mathrm{flow}^{\mathrm{cont}}(\vec{\sigma}) \,|\, \vec{\sigma} \text{ admissible stream in } \Omega\},$$
$$\Sigma^{\mathrm{stream}} = \{\vec{\sigma} \,|\, \vec{\sigma} \text{ admissible stream in } \Omega,\ \mathrm{flow}^{\mathrm{cont}}(\vec{\sigma}) = \phi^{\mathrm{stream}}\}.$$

Laws of large numbers for $\vec{\mu}_n^{\max}$ and $\mathcal{E}_n^{\min}$

Hypotheses : $(\Omega, \Gamma^1, \Gamma^2)$ is "nice" and the capacities are bounded.

Convergence of the maximal streams :

$\left(\frac{\vec{\mu}_n^{\max}}{n^d}\right)_{n\geq 1}$ converges weakly a.s. towards Σ^{stream}, i.e.,

$$a.s.\,,\ \forall f \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R})\,,\ \inf_{\vec{\sigma}\in\Sigma^{\mathrm{stream}}} \left| \int_{\mathbb{R}^d} f\, \frac{d\vec{\mu}_n^{\max}}{n^d} - \int_{\mathbb{R}^d} f\, \vec{\sigma}\, d\mathcal{L}^d \right| \xrightarrow[n\to\infty]{} 0\,.$$

Convergence of the minimal cutsets :

If $\mathbb{P}[t(e) = 0] < 1 - p_c(d) \quad (\iff \nu \neq 0)$,
$(\mathcal{E}_n^{\min})_{n\geq 1}$ converges a.s. towards Σ^{cutset}, i.e.,

$$a.s.\,,\ \inf_{F\in\Sigma^{\mathrm{cutset}}} \mathcal{L}^d(\mathcal{E}_n^{\min} \triangle F) \xrightarrow[n\to\infty]{} 0\,.$$

Continuous max-flow min-cut theorem and LLN for ϕ_n

Continuous max-flow min-cut theorem :

- $\phi^{\text{cutset}} = \phi^{\text{stream}} := \phi$,
- Σ^{cutset} and Σ^{stream} are not empty.

Convergence of the maximal flows :

$$a.s.\,, \quad \frac{\phi_n}{n^{d-1}} \xrightarrow[n\to\infty]{} \phi\,.$$

Hypotheses on $(\Omega, \Gamma^1, \Gamma^2)$

$(\Omega, \Gamma^1, \Gamma^2)$ "nice" means :

- Ω is open, bounded and connected,
- Ω is a Lipschitz domain,
- $\partial\Omega$ is included in a finite number of oriented hypersurfaces of class $\mathcal{C}^1$ that intersect each other transversally,
- Γ^1 and Γ^2 are open in $\partial\Omega$,
- $d(\Gamma^1, \Gamma^2) > 0$,
- $\mathcal{H}^{d-1}(\partial_\Gamma \Gamma^1) = \mathcal{H}^{d-1}(\partial_\Gamma \Gamma^2) = 0$.

Steps of the proof of the capacity constraint

Suppose that $\vec{\mu}_n^{\max} \rightharpoonup \vec{\sigma}\mathcal{L}^d$. Let $x \in \Omega$, $\vec{v} \in \mathbb{S}^{d-1}$, $B(\vec{v})$ a cylinder of sidelengths $1, ..., 1, h$ oriented towards $\vec{v}$.

- Lebesgue differentiation Theorem : let $B(x,\varepsilon) = x + \varepsilon B(\vec{v})$.
  $\frac{1}{\mathcal{L}^d(B(x,\varepsilon))} \int_{B(x,\varepsilon)} \vec{\sigma} \cdot \vec{v} \, d\mathcal{L}^d \underset{\varepsilon \to 0}{\longrightarrow} \vec{\sigma}(x) \cdot \vec{v} \quad \mathcal{L}^d\text{-a.e.}$
- $[\vec{\mu}_n^{\max} \rightharpoonup \vec{\sigma}\mathcal{L}^d] \Longrightarrow \left[\int_{B(x,\varepsilon)} d\vec{\mu}_n^{\max} \cdot \vec{v} \underset{n \to \infty}{\longrightarrow} \int_{B(x,\varepsilon)} \vec{\sigma} \cdot \vec{v} \, d\mathcal{L}^d \right]$.
- $\int_{B(x,\varepsilon)} d\vec{\mu}_n^{\max} \cdot \vec{v} \approx \varepsilon h n \, \mathrm{flow}_n^{\mathrm{disc}}(\vec{\mu}_n \text{ in } B(x,\vec{v}))$ $(*)$
  $\leq \varepsilon h n \, \tau_n(B(x,\varepsilon))$ by maximality of τ.
- $\frac{\tau_n(B(x,\varepsilon))}{\varepsilon^{d-1} n^{d-1}} \underset{n \to \infty}{\longrightarrow} \nu(\vec{v})$ a.s.