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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: aseceleanu2@math.unl.edu

Speaker's Name: Karen Smith

Talk Title: Introduction to Frobenius Splitting #2

Date: 08/29/12 Time: 11:30 am / pm (circle one)

List 6-12 key words for the talk: geometric Frobenius splitting

Please summarize the lecture in 5 or fewer sentances: This lecture presents Frobenius splitting from a global viewpoint, with applications to cohomology vanishing.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
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- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
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Karen Smith - Introduction to Frobenius splitting II

Characteristic 0 algebras: $R = \dfrac{\mathbb{C}[x_1, \dots, x_N]}{(f_1, \dots, f_r)}$ ← assume f_i have integer coeffs

$$R_{\mathbb{Z}} = \frac{\mathbb{Z}[x_1, \dots, x_N]}{(f_1, \dots, f_r)}$$

<u>Def</u> R is Frobenius split type (or F-regular type) if $\mathbb{F}_p \otimes_{\mathbb{Z}} R_{\mathbb{Z}} = \dfrac{\mathbb{F}_p[x_1, \dots, x_N]}{(f_1, \dots, f_r)}$ is F-split (or F-regular) for infinitely many p.

<u>Ex</u> $\dfrac{\mathbb{C}[x,y,z]}{(y^2 - zx)}$ is F-regular type

$\dfrac{\mathbb{C}[x,y,z]}{x^3+y^3+z^3}$ is F-split type because the corresponding char p elliptic curve is F-split if $p \equiv 1 \pmod 3$

<u>Thm</u> If R is F-regular (type) char p, then R is normal and CM.
<u>Cor</u>: the Hochster-Roberts Thm.

$R \longrightarrow R^{1/p^e}$ as R-module

<u>Def</u> R is <u>F-regular</u> if $\forall c \neq 0$ $\exists e$ s.t. $c^{1/p^e} R \subseteq R^{1/p^e}$ splits as R-module map
(Equivalently: $\exists\, \varphi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ s.t. $\varphi(c^{1/p^e}) = 1$)

<u>Proof of normality</u> Assume R F-regular
$R \hookrightarrow \tilde{R}$ normalization (finitely gen. R-module)
$\Rightarrow \exists\, 0 \neq c \in R$ s.t. $c\tilde{R} \subseteq R$ (e.g. the common denominator of the generators of R)
Take $\frac{x}{y} \in \tilde{R} \Rightarrow \left(\frac{x}{y}\right)^{p^e} \in \tilde{R} \Rightarrow c\left(\frac{x}{y}\right)^{p^e} \in R$

$\Rightarrow c x^{p^e} = \underbrace{r_e}_{\text{some } r_e \in R} y^{p^e}$

$c^{1/p^e} x = r_e^{1/p^e} y$ in R^{1/p^e} $\forall e$
We can find $\varphi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, $\varphi(c^{1/p^e}) = 1$

Apply φ to $c^{1/p^e}x = y r_e^{1/p^e} \Rightarrow 1 \cdot x = y \underbrace{\varphi(r_e^{1/p^e})}_{\in R} \Rightarrow \frac{x}{y} \in R$

<u>Proof of CM</u>

WLOG R complete local (can think about R graded too)

$x_1, \dots, x_d$ s.o.p.

$A = k[[x_1, \dots, x_d]] \hookrightarrow R$ finitely gen. A-mod (Cohen structure / Noether normaliz.)

Want to show $x_1, \dots, x_d$ is a regular sequence

$x_{i+1} z \in (x_1, \dots, x_i) \Rightarrow z \in (x_1, \dots, x_i)$

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & R \\ A[z] = B \arrow[ur, dashed] & M \cong B^{\oplus r} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$M \cong B^{\oplus r} \leftarrow$ maximal rank free B-submodule of R

$A[z] = B$ ... $R[x_1, \dots x_d, z]$

$\Rightarrow \exists\, 0 \neq c \in B$ s.t. $cR \subseteq M$

If $x_{i+1} z \in (x_1, \dots, x_i) R$

$x_{i+1}^{p^e} z^{p^e} \in (x_1^{p^e}, \dots, x_i^{p^e}) R$

$\Rightarrow c\, x_{i+1}^{p^e} z^{p^e} \in (x_1^{p^e}, \dots, x_i^{p^e}) B$

Because B is CM $\Rightarrow c z^{p^e} \in (x_1^{p^e}, \dots, x_i^{p^e})\ \forall e$

$\Rightarrow c^{1/p^e} z \in (x_1, \dots, x_i) R^{1/p^e} \xrightarrow{\text{apply } \varphi} z \in (x_1, \dots, x_i) R$

Find $\varphi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, $\varphi(c^{1/p^e}) = 1$

II <u>Global Point of View</u>

$$\begin{array}{ccc} R & \xrightarrow{F} & R \\ r & \longmapsto & r^p \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \operatorname{Spec} R & \longleftarrow & \operatorname{Spec} R \\ F^{-1}(p) & \longleftarrow & p \end{array}$$

is the identity map on topological space

Def If X scheme $/ k = k^p$, the Frobenius map is $X \xrightarrow{F} X$ is the identity map on underlying top. space and $\mathcal{O}_X \longrightarrow F_* \mathcal{O}_X$ is the p^{th} power map.

$r \longmapsto r^p$

Def X is Frobenius split if $\mathcal{O}_X \to F_* \mathcal{O}_X$ splits as a map of $\mathcal{O}_X$-modules.

<u>Note</u> X = affine, this definition is the same as before.

Caution: if X is projective, the existence of a global splitting of $\mathcal{O}_X \hookrightarrow F_* \mathcal{O}_X$ cannot be checked locally on stalks

e.g. X smooth $\mathcal{O}_X \hookrightarrow F_* \mathcal{O}_X$ splits locally on stalks, but usually does not split globally.

Thm: X projective, X F-split $\iff \exists \mathcal{L}$ ample invertible sheaf s.t. $S = \bigoplus_{n \in \mathbb{N}} H^0(X, \mathcal{L}^n)$ is F-split

$\iff \forall \mathcal{L}$ invertible, $S_{\mathcal{L}}$ is F-split.

Thm 2 X = Frobenius split projective variety

Take any $\mathcal{L}$ s.t. $H^i(X, \mathcal{L}^N) = 0$ $\forall N \gg 0$ (eg $\mathcal{L}$=ample)

Then $H^i(X, \mathcal{L}) = 0$.

Cor

$H^i(X, \mathcal{L}) = 0$ $\forall i > 0$, $\forall \mathcal{L}$=ample on any F-split variety X.

Cor (Kodaira Vanishing)

If X is F-split, smooth projective, then $H^i(X, \omega_X \otimes \mathcal{L}) = 0$ $\forall i > 0$ $\forall \mathcal{L}$ ample

Proof of Thm 2

$$\mathcal{O}_X \xhookrightarrow{\oplus} F^e_* \mathcal{O}_X \text{ splits } \overset{\otimes \mathcal{L}}{\Longrightarrow} \mathcal{L} \xhookrightarrow{\oplus} F^e_* \mathcal{O}_X \otimes \mathcal{L} = F_* F^* \mathcal{L} = F_* \mathcal{L}^{p^e}$$

$$\Rightarrow H^i(X, \mathcal{L}) \xhookrightarrow{\oplus} H^i(X, F_* \mathcal{L}^{p^e}) = \underbrace{H^i(X, \mathcal{L}^{p^e})}_{\substack{\| \\ 0 \text{ by assumption}}}$$

$\Rightarrow H^i(X, \mathcal{L}) = 0$ q.e.d.

Ex X = smooth curve (projective)

If $g \geq 2$ ω_X is ample $\Rightarrow H^1(X, \omega_X) \cong k$ so X not F-split.

$g = 0$ $\Rightarrow \mathcal{O}_X = \mathbb{F}_p[x, y]$ is F-split

$g = 1$ F split $\iff$ X is an ordinary elliptic curve

III DEFINITION

X is globally F-regular if $\forall$ effective (Cartier) divisor Δ $\exists e$ s.t. $\mathcal{O}_X \hookrightarrow F^e_* \mathcal{O}_X \hookrightarrow F^e_* \mathcal{O}_X(\Delta)$ splits as $\mathcal{O}_X$-mod.

<u>Thm 1'</u> X projective is globally F-regular $\iff$ some (equivalently every) section ring $S_{\mathcal{L}}$ w.r.t. $\mathcal{L}$ ample is F-regular.

<u>Thm 2'</u> X globally F-regular projective variety. Take any NEF $\mathcal{L} \Rightarrow H^1(X, \mathcal{L}) = 0$.

<u>Question</u> If X is globally F-reg, is the Picard gp f.gen? Is the Cox ring finitely generated?

<u>Observe</u> a splitting of $\mathcal{O}_X \hookrightarrow F_*^e \mathcal{O}_X$ is a map $\varphi: F_*^e \mathcal{O}_X \to \mathcal{O}_X$. Can view this as a global section of $\mathcal{H}om_{\mathcal{O}_X}(F_*^e \mathcal{O}_X, \mathcal{O}_X)$.

<u>Lemma</u> - $\mathcal{H}om_{\mathcal{O}_X}(F_*^e \mathcal{O}_X, \mathcal{O}_X) \cong F_*^e \omega_X^{1-p^e}$

If $\varphi \in \Gamma(X, F_*^e \omega_X^{1-p^e})$ is a splitting then

$\Gamma(X, F_*^e \omega_X^{1-p^e}) = H^0(X, \mathcal{O}_X((1-p^e)K_X))$.

<u>Fact</u> can be shown that if ω_X^{-1} is ample then X is globally F-regular. (i.e. X Fano $\Rightarrow X$ globally F-reg.)

<u>Thm</u> (Schwede-Smith) If X is a globally F-regular (resp F-split) projective variety, then X is log-Fano (log Calabi-Yau) i.e. $\exists$ effective $\mathbb{Q}$-divisor Δ s.t. • (X, Δ) has Kawamata log-terminal singularities (log-terminal)

• $(-K_X - \Delta)$ is ample (resp trivial)