Automne 1949

The audacity of Grothendieck's definition is to accept that *every* commutative ring A (with unit) has a scheme Spec(A) . . . This has a price. The points of Spec(A) (prime ideals of A) have no geometric sense ready to hand . . . When one needs to construct a scheme one generally does not begin by constructing the set of points. —Deligne

The crucial thing here, from the viewpoint of the Weil conjectures, is that the new notion [of space] is vast enough, that we can associate to each scheme a "generalized space" or "topos" (called the "étale topos" of the scheme in question). Certain "cohomology invariants" of this topos ("infantile" in their simplicity!) seemed to have a good chance of offering "what it takes" to give the conjectures their full meaning, and (who knows!) perhaps to give the means of proving them.

Récoltes et Semailles

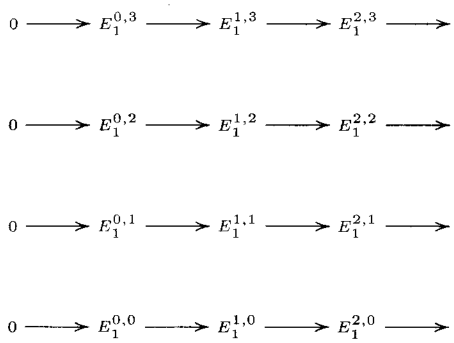
1. Find the natural world for the problem (e.g. the étale topos of a arithmetic scheme).

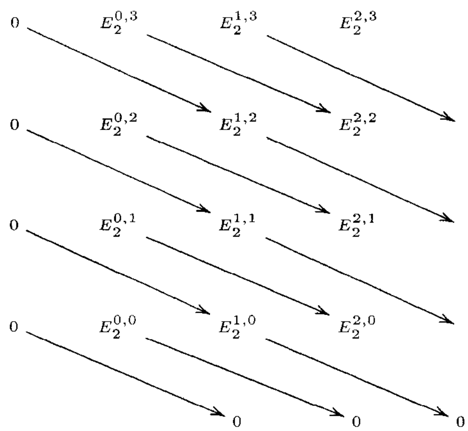
2. Express your problem cohomologically (e.g. state Weil's conjectures as a Lefschetz fixed point theorem).

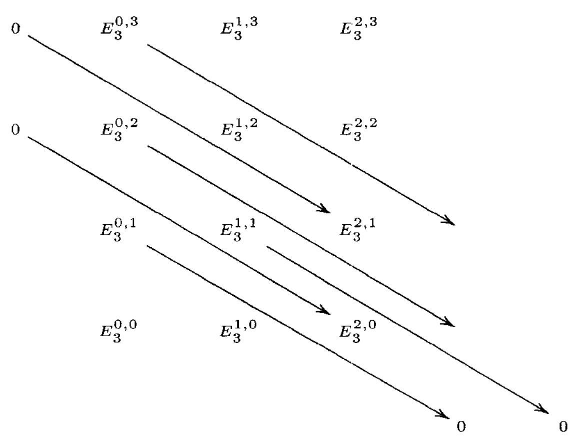
3. The cohomology of that world solves your problem, like the bursting of a ripe avocado.

A “space in the *nouveau style*” (or *topos*), generalizing traditional topological spaces, is given by a “category” which, without necessarily coming from an ordinary space, nonetheless has all the good properties (explicitly designated once and for all, of course) of such a “category of sheaves”.

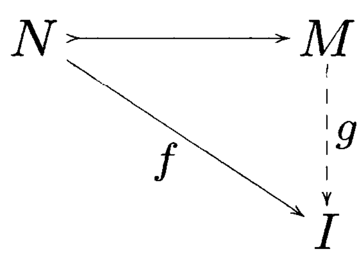
(*Récoltes et Semailles*, Promenade 39)







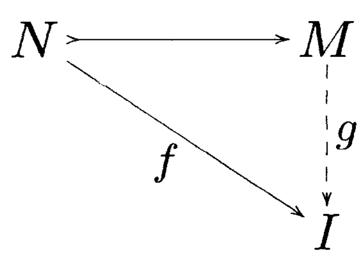
An object I is injective if: For every monic $N \rightarrowtail M$ and arrow $f:N \to I$ there is at least one $g:M \to I$ such that



Consider the set of all sheaves on a given topological space or, if you like, the prodigious arsenal of all the "meter sticks" that measure it. We consider this "set" or "arsenal" as equipped with its most evident structure, the way it appears so to speak "right in front of your nose"; that is what we call the structure of a "category"... From here on, this kind of "measuring superstructure" called the "category of sheaves" will be taken as "incarnating" what is most essential to that space. — *Récoltes et Semailles*

During the last decade the methods of algebraic topology have invaded extensively the domain of pure algebra, and initiated a number of internal revolutions. The purpose of this book is to present a unified account of these developments and to lay the foundations for a full-fledged theory. (page v)

A module I is injective if: For every module inclusion $N \rightarrowtail M$ and homomorphism $f: N \to I$ there is at least one $g: M \to I$ such that



A sheaf of Abelian groups $\mathcal{F}$ on a space X is *fine* if: For every locally finite cover of X by opens U^i there are endomorphisms ℓ^i such that:

1. for each i, the endomorphism ℓ^i is zero outside of some closed set contained in U^i.

2. the sum $\sum_i \ell^i$ is the identity

exp 15.

For each sheaf $\mathcal{F}$ on X a series of Abelian groups $\mathsf{H}^n\mathcal{F}$ and for each map $f:\mathcal{F}\to\mathcal{F}'$ a series of homomorphisms

$$\mathsf{H}^n f:\mathsf{H}^n\mathcal{F}\to\mathsf{H}^n\mathcal{F}'$$

$\mathsf{H}^n\mathcal{F}=0$ for $n>0$ and $\mathcal{F}$ fine.

For each G module M a series of Abelian groups H^nM and for each homomorphism $f:M\to M$ a series of homomorphisms

$$\mathsf{H}^n f:\mathsf{H}^nM\to\mathsf{H}^nM$$

$\mathsf{H}^nM=0$ for $n>0$ and M injective.

Betti Numbers

Let $B_0, B_1, \ldots B_{2n}$ be the Betti numbers of the complex manifold defined by the same polynomials. Then each P_k has degree B_k. And E is the Euler number, the alternating sum

$$\sum_{k=0}^{2n} (-1)^k B_k$$

Rationality:

$Z(t)$ is a rational function $\frac{P(t)}{Q(t)}$

Functional Equation:

$$Z(\frac{1}{q^n t}) \;=\; \pm q^{nE/2} t^E Z(t)$$

Riemann Hypothesis

$$Z(t) \;=\; \frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)}$$

Where each P_k is an integer polynomial with all roots of absolute value $q^{-k/2}$.

Let N_s be the number of points defined on $\mathbb{F}_{(q^s)}$.

$$Z(t) = \exp\,(\sum_{s=1}^{\infty} N_s \frac{t^s}{s})$$

We would be badly blocked if there were no bridge between the two.

And *voilà* god carries the day against the devil: this bridge exists; it is the theory of algebraic function fields over a finite field of constants. (Letter of 26 March 1940)

In this "swelling sea", the theorem is "submerged and dissolved by some more or less vast theory, going well beyond the results that were originally to be established"

(*Récoltes et Semailles* 555)

A different image came to me a few weeks ago. The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration. . . the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it. . . yet it finally surrounds the resistant substance.

(*Récoltes et Semailles* 552-3)

Grothendieck himself can't necessarily be blamed for this [category theory] since his own use of categories was very successful in solving problems.

—Miles Reid

Tout ce qu'il y a de bébête!

(infantile in its simplicity)

La mer qui monte