Lagrangian tori in non-selfadjoint spectral theory

with J. Sjöstrand

Work on $\mathbb{R}^2$, p - holomorphic $f^{\underline{n}}$ in a tubular nbhd. of $\mathbb{R}^4 \subset \mathbb{C}^4$ s.t.

$$p(x,\xi) = \mathcal{O}(\underbrace{m(\mathrm{Re}(x,\xi))}_{\text{order } f^{\underline{n}}})$$

$$1 \leq m(X) \leq C \langle X-Y \rangle^{N_0} m(Y)$$

<u>Will take</u>: $m = 1$

* $|p(x,\xi)| \geq 1/C$, $|(x,\xi)| \geq C$, $C > 0$

* $p|_{\mathbb{R}^4}$ is real, $dp \neq 0$ on $\underbrace{p^{-1}(0) \cap \mathbb{R}^4}_{\text{connected}} \Subset \mathbb{R}^4$

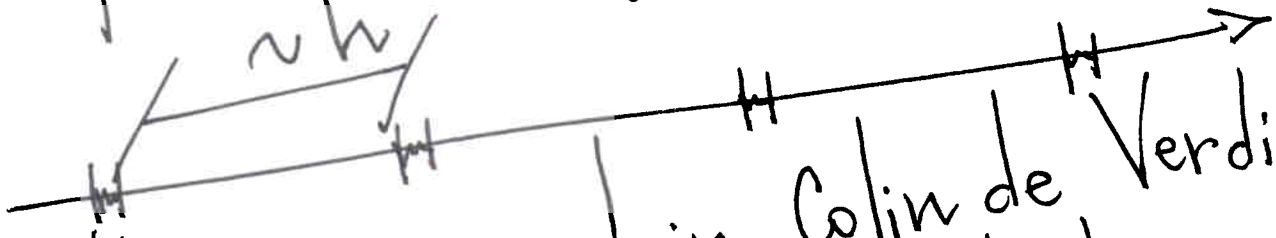
(H) The H_p-flow is periodic on $p^{-1}(E) \cap \mathbb{R}^4$ with minimal period $T(E) > 0$, analytic in $E \in \mathrm{neigh}(0, \mathbb{R})$.

Quantum level:

$$P = P(x, hD_x; h) \sim \sum_{j=0}^{\infty} h^j p_j^w(x, hD_x)$$

selfadjoint with $p_0 = p$

$\mathrm{Spec}(P) \cap \mathrm{neigh}(0, \mathbb{R})$:



$O(h^2)$ Weinstein, Colin de Verdière...

(require that p_1 is constant on $p^{-1}(E)$ $\forall E \sim 0$)

Non-selfadjoint family:

$$P_\varepsilon = P(x, hD_x, \varepsilon; h) \sim \sum_{j=0}^{\infty} h^j p_j^w(x, hD_x, \varepsilon)$$

$\varepsilon \sim 0$ in $\mathbb{R}$, $P_0 = P$ and

$$p_0(x, \xi, \varepsilon) = p + i\varepsilon q + \varepsilon^2 r + O(\varepsilon^3)$$

Understand: Spec(P_ε) near 0 in $\mathbb{C}$

(discrete!)

Set: $\langle q \rangle = \frac{1}{T(E)} \int_0^{T(E)} q \circ \exp(tH_p)\,dt$ on $p^{-1}(E)$

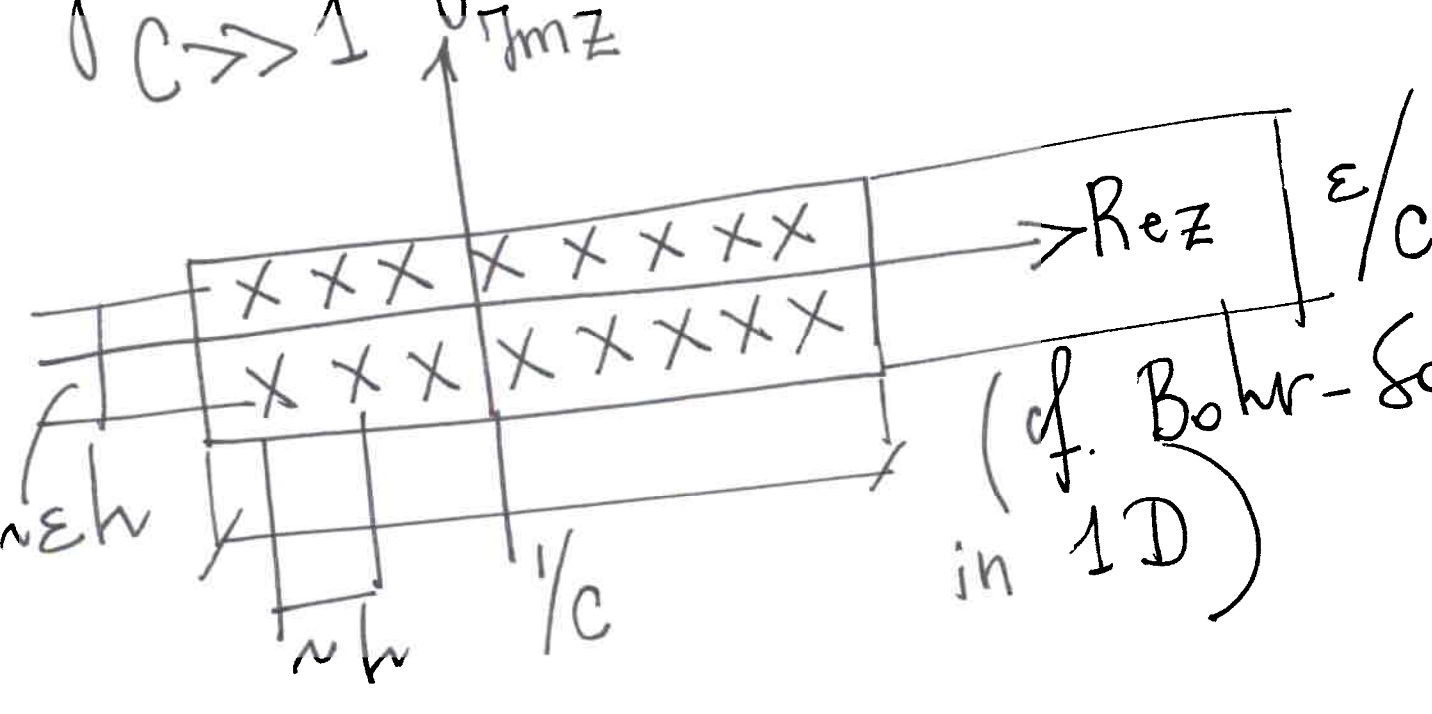
$\Rightarrow H_p \langle q \rangle = 0.$

If $(0,0)$ is a regular value of

$\mathbb{R}^4 \ni \rho \longmapsto (p(\rho), \operatorname{Re}\langle q \rangle(\rho)) \in \mathbb{R}^2$,

get all eigenvalues of P_ε in $[-1/C, 1/C] \times \varepsilon[-1/C, 1/C]$,

$C \gg 1$



(cf. Bohr-Somm. in 1D)

provided $h \ll \varepsilon = \mathcal{O}(h^\delta), \delta > 0.$

(Sjöstrand - H. '03)

This talk: What happens when $\langle q \rangle \equiv 0$?

Example $P_\varepsilon = -h^2 \Delta + i\varepsilon q(x)$ on S^2

q is odd

Radon: $q \longmapsto \langle q \rangle$

$C^\infty(S^2) \longrightarrow C^\infty(S^2)$

Selfadjoint case: Guillemin, Uribe, Ivrii...

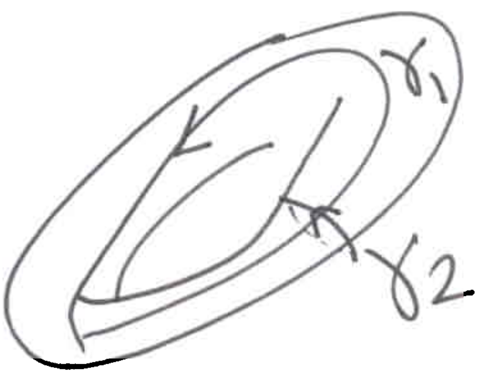
Friedlander: $-\Delta + q, \; q \in L^\infty(S^2, \mathbb{C})$

(~1980)

Set: $s = r + \frac{1}{2T} \int_0^T t \{ q, q \circ \exp(tH_p) \} dt$

<u>Assume</u>: dp, $d\,\mathrm{Im}\langle S\rangle$ are linearly indep. on $\Lambda_{0,0}: p=0,\ \mathrm{Im}\langle S\rangle = 0 \implies$

$\Lambda_{0,0}$ is a Lagr. torus (if connected)



$$\mathcal{X}: \mathrm{neigh}(\Lambda_{0,0}, T^*\mathbb{R}^2) \longrightarrow \mathrm{neigh}(\xi = 0, T^*\mathbb{T}^2)$$

$$p \circ \mathcal{X}^{-1} = p(\xi_1)$$

$$\langle S\rangle \circ \mathcal{X}^{-1} = \langle S\rangle(\xi)$$

<u>Thm. 1</u> (Sjöstrand - H. '03)

Assume $\mathrm{Sub}(P_{\varepsilon=0}) = 0$ and $h \ll \varepsilon = \mathcal{O}(h^\delta)$, $\delta > 0$. Then $\mathrm{Spec}(P_\varepsilon)$ in $[-1/C, 1/C] + i\varepsilon^2[-1/C, 1/C]$ is ~~are~~ given by

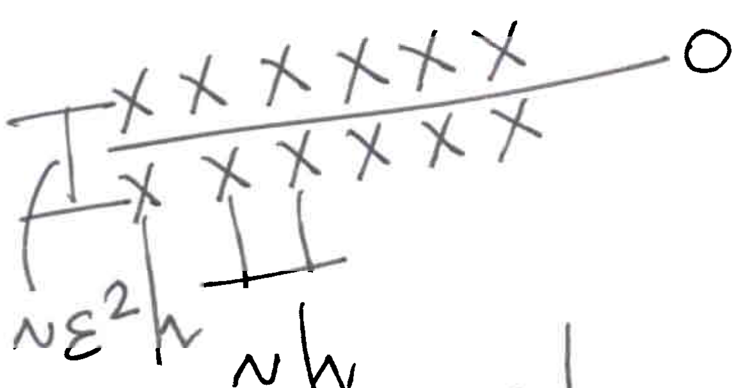
$$\hat{P}\left(h\left(k-\frac{\alpha}{4}\right)-\frac{S}{2\pi}, \varepsilon, \frac{h}{\varepsilon}; h\right) + O(h^{\infty}),$$

$k \in \mathbb{Z}^2$, with

$$\hat{P}\left(\xi, \varepsilon, \frac{h}{\varepsilon}; h\right) = p(\xi) + \varepsilon^2\left(r_0 + h r_1 + \dots\right)$$

$$r_0 = \langle S \rangle(\xi) + O\left(\varepsilon + \frac{h}{\varepsilon}\right)$$

$$r_j = O(1), \quad j \geq 1$$



Remark Get clusters if $\varepsilon^2 \ll h$.

Idea of proof

Step 1 Reduce P_ε to $p + \varepsilon^2 \langle S \rangle + O(\varepsilon^3 + \varepsilon h)$ by averaging + QBNF near $\Lambda_{0,0}$

Step 2 Justify!

Prop. $\exists$ an $\mathbb{R}$-mfld. $\Lambda \subset \mathbb{C}^4$ $(\varepsilon + h/\varepsilon)$-close to $\mathbb{R}^4$ and $= \mathbb{R}^4$ far away, and C^∞-Lagr. torus $\hat{\Lambda}_{0,0} \subset \Lambda$ s.t. away from $\hat{\Lambda}_{0,0}$, $|\mathrm{Im}\, P_\varepsilon| \geq \varepsilon^2/\mathcal{O}(1)$ or $|\mathrm{Re}\, P_\varepsilon| \geq 1/\mathcal{O}(1)$.

Also, $\exists$ elliptic h-FIO

$$U = \mathcal{O}(1) : H(\Lambda) \longrightarrow L^2_\theta(\mathbb{T}^2)$$

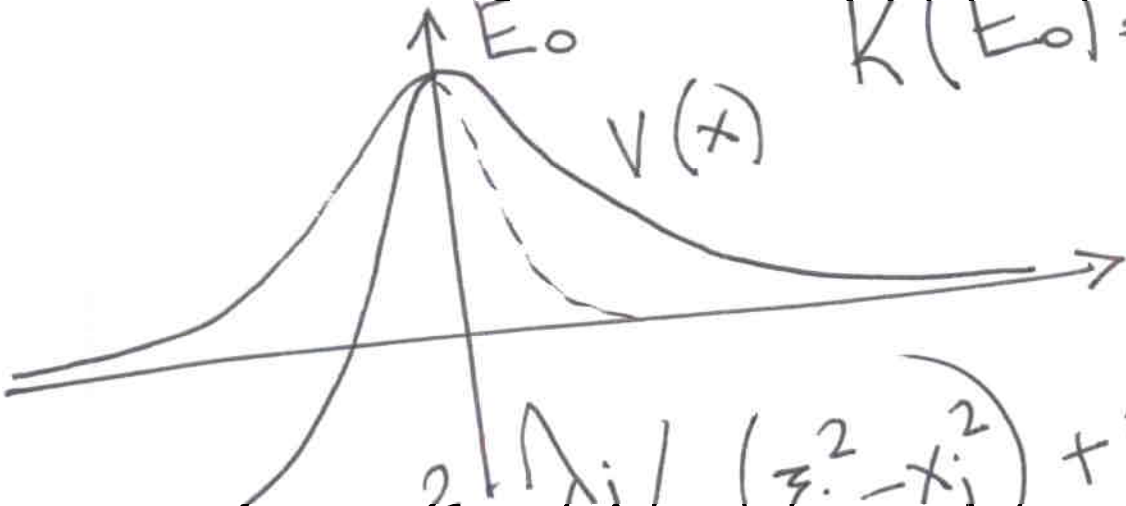
s.t. $U P_\varepsilon = \hat{P} U$ microlocally near $\hat{\Lambda}_{0,0}$.

$$\hat{P} = \hat{P}(hD_x, \varepsilon, h/\varepsilon; h)$$

Application Barrier top resonances in 2D

$P = -h^2\Delta + V$, V dilation analytic

$K(E_0) = \{(0,0)\}$



$$p(x,\xi) = \sum_{j=1}^{2} \lambda_j/2\,(\xi_j^2 - x_j^2) + p_3(x) + p_4(x) + \dots$$

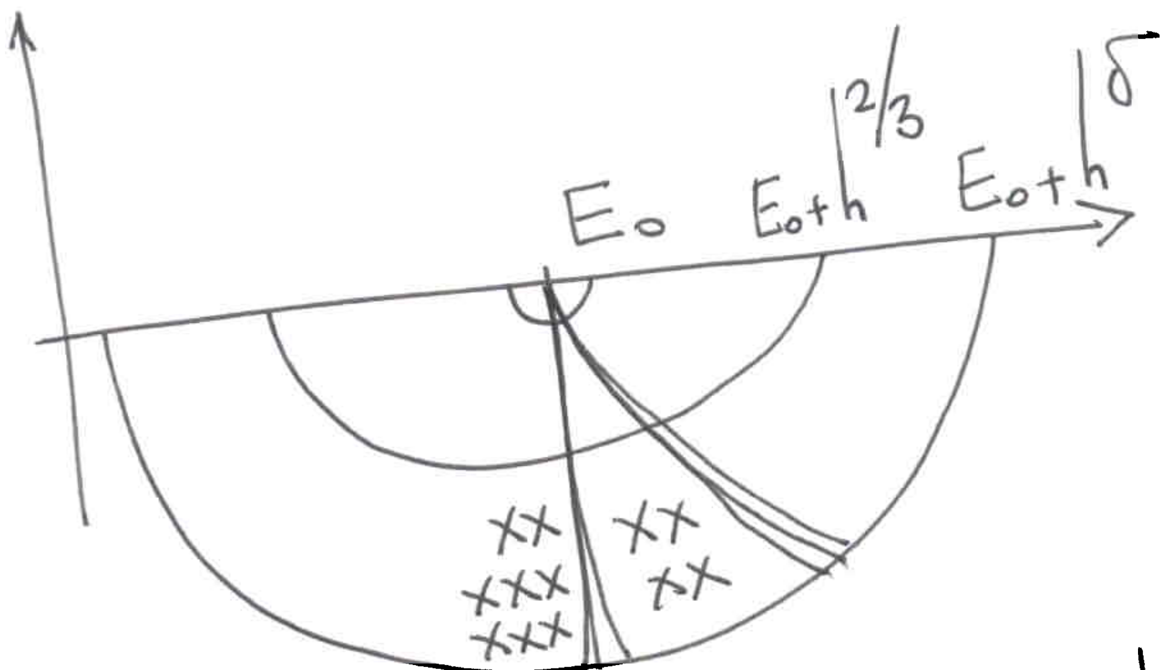
Complex scaling

$$\longrightarrow \underbrace{\sum_{j=1}^{2} \lambda_j/2\,(\xi_j^2 + x_j^2)}_{=\,p_2} + ie^{3\pi i/4} p_3(x) - ip_4(x) + \dots$$

λ_j are $\mathbb{Z}$-dependent $\Longrightarrow$ H_{p_2}-flow is periodic.

1-1 resonance: $\lambda_j = 1$, $j = 1,2$ $\Longrightarrow$

$\langle p_3 \rangle \equiv 0$ while $\langle s \rangle$ is purely imaginary, Thm. 1 applies to give all resonances in



<u>Exclude</u>

$$\bigcup_j \left\{ z_j \;\middle|\; \left| \mathrm{Re}\, z - E_0 - A_j \left|\mathrm{Im}\, z\right|^2 \right| < 1/O(1) \left|\mathrm{Im}\, z\right|^2 \right\}$$

critical values of

$$\mathrm{Im}\,\langle s \rangle \Big|_{p_2 = 1}; \quad \text{If } p_3(x) = x_1^3, \; p_4(x) \equiv 0$$

$$\langle s \rangle = \frac{15 i}{2} \left(\frac{x_1^2 + \xi_1^2}{2} \right)^2$$

Now $q|_{\mathbb{R}^4}$ is real $\Longrightarrow \langle s \rangle$ is real

By averaging reduce P_ε to

$$p + \varepsilon^2 \underbrace{\langle s \rangle}_{\text{real}} + i\varepsilon^3 \langle t \rangle + \mathcal{O}(\varepsilon^4 + \varepsilon h)$$

t cubic in q

$\Longrightarrow \operatorname{Im} z = \mathcal{O}(\varepsilon^3)$, if $z \in \operatorname{Spec}(P_\varepsilon)$, $z \sim 0$

Assume: $dp, d\langle s \rangle$ are linearly indep. on

$\Lambda_{0,0}$: $p = 0$, $\langle s \rangle = 0$.

Set: $\langle\langle t \rangle\rangle_T = 1/T \int_0^T \langle t \rangle \circ \exp(uH_{\langle s \rangle})\,du$

$T > 0$

Then $\langle\langle t \rangle\rangle_T(\rho) \xrightarrow[T \to \infty]{} \underbrace{\langle\langle t \rangle\rangle_\infty}_{\text{mean value of } \langle t \rangle \text{ over } \Lambda_{0,0}} \in \mathbb{R}$

$\rho \in \Lambda_{0,0}$

Global assumption:

(H) $\forall$ flow-inv. nbhd. W of $\Lambda_{0,0}$ in $p^{-1}(0)$ $\exists\, T_0>0,\ C>0$ s.t.
$\mathrm{dist}\left(\langle\langle t\rangle\rangle_T(\rho),\ \langle\langle t\rangle\rangle_\infty\right) \geq 1/C$, $T\geq T_0$,
$\rho \in \complement W$.

<u>Thm. 2</u> Assume (H), $\mathrm{Sub}(P_{\varepsilon=0})=0$, $\delta>0$. Then
$h^{1/2-\delta} \ll \varepsilon = \mathcal{O}(h^{\delta})$, $\mathrm{Spec}(P_\varepsilon)$
in $[-1/C, 1/C] + i\varepsilon^3\left[\langle\langle t\rangle\rangle_\infty - 1/C,\ \langle\langle t\rangle\rangle_\infty + 1/C\right]$, $C \gg 1$
is given, mod $\mathcal{O}(h^\infty)$, by the formal "quasi-eigenvalues" associated with $\Lambda_{0,0}$.