J. Herzog "Ideals of fiber type"

Joint work with T. Hibi, M. Vladoru

K = field

S = polynomial ring over K

$I = (f_1, \dots, f_m) \subset S$ ← all generators of = degree

$$R(I) = \bigoplus_{j \geq 0} I^j t^j = S[f_1 t, \dots, f_m t] \subseteq S[t]$$

(Rees Ring)

$$\varphi : T = K[x_1, \dots, x_n, y_1, \dots, y_n] \longrightarrow R(I)$$

$$x_i \longmapsto x_i$$

$P = \ker \varphi$

$$y_j \longmapsto f_j$$

~~$R(I)$~~ T is bigraded by $\deg(x_i) = (1,0)$, $\deg(y_j) = (0,1)$

$$T \longrightarrow R(I)$$

$$T \searrow S(I) \nearrow R(I)$$

$S(I)$ symmetric algebra

$S(I) = T/L$ where $L = (g_1, \dots, g_r)$,

L = "relation matrix" $\quad g_j = \sum_{i=1}^{m} \alpha_{ij} y_i$

I is <u>of linear type</u> if $S(I) \to R(I)$ is an isomorphism; then

$$R(I)/\mathfrak{m}R(I) \cong K[f_1, \dots, f_m]$$

$$= K[y_1 \dots y_m]/J \quad \text{(fiber relations)}$$

Defn I is <u>of fiber type</u> if $\overset{\ker\varphi}{\overset{\parallel}{P}} = (L, J)$.

i.e., P is generated by relations of bidegree $(*, 1)$ and $(0, *)$.

Remark If I is of fiber type and has linear relations, then I^k has linear relations $\forall k$.

Defn I is <u>of Gröbner fiber type</u> if there is a term order such that the Gröbner basis of P is generated by elements of bidegree $(*, 1)$ and $(0, *)$.

Proposition: If I is of Gröbner fiber type and has a linear resolution, then I^k has a linear resolution $\forall k$.

Examples ① $I = (x_1^2, \; x_1x_2, \; x_2^2)$

$P = (x_2y_1 - x_1y_2, \quad x_2y_2 - x_1y_3, \quad y_1y_3 - y_2^2)$ ← fiber relation

② $I = (x_1x_2x_3, \; x_2x_4x_5, \; x_5x_6x_7, \; x_3x_6x_7)$

$P = (\ldots, \; x_4y_1y_3 - x_1y_2y_4)$ is not of fiber type.

Thm (Villareal) $I = (f_1, \ldots, f_m)$ monomial ideal, (all f_i of equal degree.)

For all nondecreasing sequences $\alpha = (i_1, \ldots, i_s)$ and $\beta = (j_1, \ldots, j_s)$ for which

$$f_{i_1} \cdots f_{i_s} = f_\alpha \neq f_\beta = f_{j_1} \cdots f_{j_s},$$

there exist r, t such that

$$f_{j_t}\left(f_\alpha / f_{i_t}\right) \;\Big|\; \operatorname{lcm}(f_\alpha, f_\beta)$$

$\Rightarrow I$ is of fiber type.

E.g., this condition is satisfied $\forall$ squarefree ideals generated in degree 2. But not in degree 3 (see example above, #2).

Polymatroid Ideals

$[n] := \{1, \dots, n\}$

$\rho : 2^{[n]} \to \mathbb{Z}_+$

nondecreasing and submodular

(i.e., ① $\rho(A) \leq \rho(B)$ for $A \subseteq B \subset [n]$
and ② $\rho(A) + \rho(B) \geq \rho(A \cap B) + \rho(A \cup B)$.)

$$P = \{ u \in \mathbb{Z}_+^n \mid u(A) \leq \rho(A) \text{ for } A \in 2^{[n]} \},$$

where $u(A) := \sum_{i \in A} u_i$,

is called a discrete polymatroid.

$$B := \{ u \in P \mid u([n]) = \rho([n]) = d \}$$

$$B \subset N_d = \{ u \in \mathbb{Z}_+^n \mid u([n]) = d \},$$

the set of bases of P.

Herzog ③

$I(B) \subset K[x_1, \ldots, x_n]$ is defined by

$$I(B) = (x^u \mid u \in B)$$ (the polymatroid ideal)

and $K[B] = K[x^u \mid u \in B]$ base ring of P

$$R(I(B)) / m(R(I(B))) = K[B]$$

Thm Let $f_1, \ldots, f_m$ be generators of $I(B)$ in lex order. Then

$$(f_1, \ldots, f_{i-1}) : f_i$$

is generated by monomials of degree 1.

So $I(B)$ is said to have linear quotients $\Rightarrow$ $I(B)$ has a linear resolution.

$I(B)^k$ are polymatroid ideals $\forall k$, hence have linear resolutions.

Thm: $I(B)$ is of fiber type, and is of Gröbner fiber type provided that B is sortable.
(↑ the underlying polymatroid)

$P = (1,1)$
$(0,*)$

Conjecture (N. White): $K[B]$ .... quadratically.... generated?

This thm + White's conjecture $\Rightarrow$ $R(I(B))$ quadratically generated.

Herzog (4)

Given a term order $<$ on $K[Y]$ and a Gröbner basis G of J_B w.r.t. $<$,

$T = K[X,Y]$, and extend $<$ to T by

$$x^a y^b < x^{a'} y^{b'} \iff x^a <_{lex} x^{a'} \text{ or } x^a = x^{a'} \text{ and } y^b < y^{b'}$$

and $x_1 > x_2 > \cdots$

<u>Defn</u> A monomial $y^b \in K[Y]$ is <u>standard</u> (w.r.t. $<$) if $y^b \notin in_<(J_B)$

<u>Thm</u> Suppose for any two standard monomials

$$\prod_{r=1}^{N} y_{a_r}, \qquad \prod_{r=1}^{N} y_{b_r}$$

$$\downarrow \qquad\qquad \searrow$$

$$\prod x^{u_r} = x_1^{a_1} \cdots x_n^{a_n} \qquad \prod x^{v_r} = x_1^{b_1} \cdots x_n^{b_n}$$

$$a_1 = b_1, \ldots, a_{q-1} = b_{q-1}, \quad a_q < b_q$$

there exist $1 \le r \le N$, $q < j$ such that

$$x_q (x^{u_r}) / x_j \in I(B).$$

Then P has a Gröbner basis in degrees $(1,1)$, $(0,*)$.

Herzog

Corollary: In this case, I^k has a linear resolution $\forall k$.

Examples: ① I is strongly stable — satisfies the conditions of the theorem.

② (due to CoCoA, Conca, De Negri)

I has Borel generators $(y^6 z, \; x^2 y^2 z^3, \; x^3 z^4)$

$\mu(I) = 23$ (# of generators)

J_B = 199 quadrics and one cubic

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extend:

M = module

$S(M)$ = symmetric algebra

$R(M) = S(M) / S\text{-Torsion}$

E_i = Koszul complex