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K-theory for algebraic stacks

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Main question: Describe $K(F)$ = ring spectrum of K-theory of an algebraic stack F.

Motivation: If F - DM stack the Chern character map

$$ch : K_0(F) \to A^\circ(F)_{\mathbb{Q}}$$

has a big kernel. Moreover RR does not work for this Chern character map. So the idea will be to describe $K_0(F)$ explicitly before we look at ch.

Remark: There ~~are~~ precise conjectures of what $K_0(F)$ should be for $[X/G] = F$.

Goal: Find a formula for $K(F)_{\mathbb{Q}}$ for a regular DM stack and $G(F)_{\mathbb{Q}}$ for any DM stack.

<u>Plan</u>: (1) The stack $\mathcal{C}_F^t$

(2) The morphisms ϕ_F, ψ_F

(3) The main theorem

(4) Open questions.

1. The stack $\mathcal{C}_F^t$

Let F be a DM stack, seperated and noetherian

Let $\mathcal{C}_F : (Sch) \to (Gpds)$

$$X \to \begin{cases} \underline{ob} \quad (s, c) \quad s \in F(X) \\ \quad c \subseteq \underline{Aut}(s) \text{ - finite} \\ \quad\quad \downarrow \quad\quad \text{etale} \\ \quad\quad X \quad\quad \text{cyclic subgroup} \\ \\ \underline{Mor} \quad (s, c) \xrightarrow{\sim} (s', c') \\ \\ \quad u : s \xrightarrow{\sim} s' \quad \text{s.t.} \\ \quad \text{in } F(X) \quad\quad u c u^{-1} = c' \end{cases}$$

$\mathcal{C}_F$ is again a DM stack

$$\pi : \mathcal{C}_F \to F \quad \text{- finite unramified}$$
$$(s, c) \to s$$

We have

$$\mathcal{C}_F^t \subset \mathcal{C}_F \quad \text{- the tame part}$$

i.e.

$\mathcal{C}_F^t$ = open substack in $\mathcal{C}_F$ consisting of (S, C) s.t. order (C) is invertible on X.

Moreover $\exists$ a universal group scheme

$$\mathcal{C} \to \mathcal{C}_F^t$$

s.t.

$$\mathcal{C}_{|(S,C)} \cong C \to X$$

and the associated sheaf of characters

$$\chi := \underline{\mathrm{Hom}}(\mathcal{C}, \mathbb{G}_m)$$

+ the sheaf of group algebras

$$\begin{array}{l} \mathbb{Q}[\chi] \quad = \text{sheaf of group algebras} \\ \downarrow \\ \Lambda_F \quad = \text{maximal cyclotomic quotient} \end{array}$$

locally $\mathbb{Q}[\chi] = \dfrac{\mathbb{Q}[T]}{T^m - 1}$

$$\Lambda_F = \frac{\mathbb{Q}[T]}{\phi_m(T)} = \mathbb{Q}[\zeta_m]$$

<u>Def:</u> $\underline{K}^{\chi}(F) := \mathbb{H}_{et}(\mathcal{C}_F^{t}, \underline{K} \otimes \Lambda_F)$

$\underline{G}^{\chi}(F) := \mathbb{H}_{et}(\mathcal{C}_F^{t}, \underline{G} \otimes \Lambda_F)$

<u>Note:</u> Here $\underline{K} \otimes \Lambda_F := \underline{K} \wedge K(\Lambda_F, \cdot)$

<u>Example:</u> $F = [X/G]$ $\quad X/\mathbb{C}$

$$\underline{K}_*^{\chi}(F) \otimes \mathbb{C} \cong \bigoplus_{h \in \mathrm{conj}(G)} K_*(X^h)^{Z_h}_{\mathbb{C}}$$

(compare with Vistoli's formula)

e.g. if

$X = \mathrm{Spec}\,\mathbb{C} \rightsquigarrow \mathbb{C}(G) = K_0^{\chi}([*/G])$

2. The morphisms ϕ_F, ψ_F

We would like to compare

$K(F)_{\mathbb{Q}}$ (:= the K-theory of perfect complexes on F)

$G(F)_{\mathbb{Q}}$ (:= the K-theory of coherent sheaves on F)

with $\underline{K}^{\chi}(F)$, $\underline{G}^{\chi}(F)$.

To define $\phi_F : K(F) \to \underline{K}^{\chi}(F)$ we first look at the natural pull back map

$$\pi^* : K(F) \to K(\mathcal{E}_F^t)$$

and compose it by the diagonalization map

$$d : K(\mathcal{E}_F^t) \to \underline{K}^{\chi}(F)$$

defined as follows.

Given a vector bundle V on $\mathcal{E}_F^t$ $\Rightarrow$ $\exists$ a natural action of $\mathcal{E}$ on V (since $\mathcal{E} \subset$ Inertia stack of F)

In particular locally on $\mathcal{E}_F^t$ $\Rightarrow$ the action of $\mathcal{E}$ on V can be diagonalized and so

$$V \cong \bigoplus_{\rho \in \chi} V^{\rho}$$

$\Rightarrow$ put $d(V) = \sum_{\rho \in \chi} [V^{\rho}]\rho$.

Remarks: • ϕ_F is a ring morphism which is contravariant

• $\exists\ \psi_F : G(F)_{\mathbb{Q}} \to \underline{G}^{\alpha}(F)$ which is a module over ϕ_F and is covariant for proper maps of finite cohomological dimension.

3. The main theorem

Thm 1) ψ_F is an equivalence

2) if F is regular then ϕ_F is an equivalence

3) If F is smooth over a regular scheme S and F has a quasiprojective moduli space over S, then $\exists\ \alpha_F \in \underline{K}_0^{\alpha}(F)$ s.t.

$$\alpha_F^{-1} \cdot \phi_F(-) = \psi_F$$

In particular $K(F) \otimes \mathbb{Q} = G(F) \otimes \mathbb{Q}$.

Remark: • 1) + 3) is just Lefschetz–Riemann–Roch.

- For the proof: use étale descent over the moduli space of F and reduce the question to $[X/G]$ where one can apply Vistoli's formula.

- If $F = [X/GL_n]$, X - smooth over a field k $\Rightarrow$
$$K(F)_{\mathbb{Q}} \cong \underline{K}^{\chi}(F)$$
is equivalent to the Vistoli-Vezzosi formula.

- If we have $k = \bar{k}$ and $f : F \to F'$ is a proper map of finite coho dimension between smooth stacks $/k$ $\Rightarrow$

$$\begin{tikzcd}
G_*(F)_{\bar{\mathbb{Q}}} \arrow[r, "f_*"] \arrow[d, phantom, "\cong" rotate=90] & G_*(F')_{\bar{\mathbb{Q}}} \arrow[d, "\cong"] \\
\underline{G}^{\chi}_*(F)_{\bar{\mathbb{Q}}} \arrow[r, "f_*"'] \arrow[d, phantom, "\cong" rotate=90] & \underline{G}^{\chi}_*(F)_{\bar{\mathbb{Q}}} \arrow[d, phantom, "\cong" rotate=90] \\
G_*(M_{I_F^t})_{\bar{\mathbb{Q}}} \arrow[r, "f_*"'] \arrow[d] & G_*(M_{I_{F'}^t})_{\bar{\mathbb{Q}}} \arrow[d] \\
H_*(M_{I_F^t}) \arrow[r, "\text{usual RR on alg. space}", "f_*"'] & H_*(M_{I_{F'}^t})
\end{tikzcd}$$

<u>Note</u>: Here $I_F^t \subset I_F$ - tame inertia stack of F and $M_{I_F^t}$ - the coarse moduli

space.

Corollary: $$G_*(F)_{\bar{\mathbb{Q}}} \cong K_*(F)_{\bar{\mathbb{Q}}} \xrightarrow{\ \sim\ } A^\cdot(I_F^t, *)_{\bar{\mathbb{Q}}}$$

for F regular over a field with quasi-projective moduli space.

4. Open questions

1) The definition of $\mathcal{C}_F^t$ and $\underline{K}^\chi(F)$ and the map ϕ_F make sense for Artin stacks. One does not expect this to be an iso in general.

Question: If F smooth with finite diagonal is it true that ϕ_F is an isomorphism?

Remark: The Vezzosi-Vistoli formula implies that the answer is 'yes' if F is a quotient stack

Note: Here $\mathcal{C}_F^t = (\xi, C)$, $C \subset \mathrm{Aut}$ - cyclic subgroup of multiplicative type.

2) Going back to the DM case

$\Rightarrow$

$$\phi_F : K(F)_{\mathbb{Q}} \to \underline{K}^{\chi}(F)_{\mathbb{Q}}$$

$$\cong$$

$$\mathbb{H}_{et}(\mathcal{C}_F^t, \underline{K} \otimes \Lambda_F)$$

λ-ring

$\Rightarrow$ can pull back the γ-filtration to get a γ-filtration on $K_*(F)$.

<u>Question</u>: How can one describe this filtration directly?

<u>Note</u>: This pullback γ-filtration is not the usual one.

3) If char $k = 0$. $\exists$ notion of cyclic homology and periodic cyclic homology of stacks.

$HC(F)$, $HP(F)$ can be defined similarly to K and G (e.g. Keller's construction)

Moreover it is known that

$$HP(BG) \cong k(G)$$

<u>Question</u>: Describe HC and HP of a DM stack F. Is it true that

$$HP(F) \cong H^*_{DR}(I_F)$$

for smooth F?

If so one can define Chow groups of Artin stacks in this way.

4) K-theory of higher stacks
Using Simpson's theory of n-geometric stacks.

If F - 2-DM stack i.e.

$$F \leftarrow F_0 \overset{s}{\underset{t}{\leftleftarrows}} F_1 \qquad F_0, F_1 \text{ - DM stacks}$$
$$s, t \text{ - etale.}$$

<u>Observation</u>: Every such F is always a 2-gerbe over a 1-DM stack i.e.

$$F \to \tau_{\leq 1} F \quad \text{has fiber } K(H, 2)$$
$$H \text{ - finite group}$$

$\Rightarrow \quad K(F)_{\mathbb{Q}} \simeq K(\tau_{\leq 1} F)_{\mathbb{Q}}.$

<u>Note</u>: If F n-geometric Artin stack $\Rightarrow$

$$K(F) \simeq K(\tau_{\leq 1} F)$$

but $\tau_{\leq 1} F$ may not be algebraic.

Idea: use higher vector bundles on F.

<u>Example</u>: • A 2-vector bundle on F is a stack of 1-categories $\mathcal{E}$ which is a module over the stack <u>Vect</u> of vector bundles.

• $2\text{-Pic}(F) \simeq H^2(F, \mathbb{G}_m)$

e.g.

$$2\text{-Pic}(K(H,2)) \simeq \mathrm{Hom}(H, \mathbb{G}_m)$$

i.e. the K-theory of 2-vector bundles is <u>not trivial</u>.

It can be nasty!

$$K(2\text{-Vect}(k)) \simeq K(K(k))$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \shortparallel$

K-theory of the ring spectrum $K(k)$.