DG-schemes in alg. geometry

$$\text{"Derived category" of schemes} = ? \xleftarrow{\text{der. functors}} \text{usual schemes}$$

left $\sim \mathcal{D}^{\leq 0}$ right $\sim \mathcal{D}^{\geq 0}$

Functors:

$\varinjlim$: quotients

e.g. $X/R = \varinjlim \{R \overset{pr}{\underset{p_2}{\rightrightarrows}} X\}$
eq. rel

right exact (= comm. with other $\varinjlim$)

Need $L\varinjlim$ or $\underrightarrow{\mathrm{holim}}$

stacks, n-stacks ..

$\mathcal{D}^{[-1,0]}(Sch)$ $\mathcal{D}^{[-n,0]}Sch$

X/G as a stack $= \underrightarrow{\mathrm{holim}}(G \times X \overset{pr}{\underset{action}{\rightrightarrows}} X)$

$\varprojlim$: defining varieties by equations conditions

e.g. $\varprojlim\{X \overset{f}{\underset{g}{\rightrightarrows}} Y\} = \{x : f(x) = g(x)\}$

left exact (= commutes with other $\varprojlim$)

Need $R\varprojlim$ or $\underleftarrow{\mathrm{holim}}$

$\mathcal{D}^{\geq 0}(\text{Aff. Schemes}/\mathbb{C}) =$

$= \mathcal{D}^{\leq 0}(\text{comm. algebras})$

$\downarrow$

dg-algebras in degrees ≤ 0

$\{ \cdots \to A^{-1} \to A^0 \} = A^{\bullet}$

$d(A^0) = 0$ d A^0-linear

$\Rightarrow$ get a sheaf $\mathcal{O}^{\bullet}$ on $\mathrm{Spec}(A^0)$

$\mathrm{Spec}(A) = (\mathrm{Spec}(A^0), \mathcal{O}^{\bullet}, d)$
dg-ringed space

$\mathbb{Z}_{\le 0}$-graded

<u>Def.</u> A dg-scheme = a dg-ringed space $X = (X^0, \mathcal{O}_X^\bullet, d)$ with $\mathcal{O}^0_X = \mathcal{O}_{X^0}$, locally $\simeq \operatorname{Spec}(A^\bullet)$

<u>Ex.</u> $V^\bullet = \{V^0 \to V^1 \to \dots\}$ complex of fin.-dim. vector spaces

$$P(V^\bullet) = (P(V^0), \mathcal{O}^\bullet)$$

$$\mathcal{O}^\bullet(U) = \Big(\underset{\mathbb{Z}_-\text{-graded}}{S(V^*)} \underset{\substack{S(V^{0*}) \\ \| \\ \mathbb{C}[V]}}{\otimes} \text{Rat. f. on } V^0 \text{ regular over preimage of } U \Big)$$

degree 0 w.r.t. grading coming from $S^\bullet$

(Also Grassmannians etc.)

$$\mathcal{O}_{X^0} = \mathcal{O}^0_X \longrightarrow \underline{H}^0(\mathcal{O}_X^\bullet) = \mathcal{O}^0_X / d\,\mathcal{O}^{-1}_X$$

$$X_0 \hookleftarrow \operatorname{Spec}(\underline{H}^0(\mathcal{O}^\bullet)) =: \pi_0(X)$$

<u>Smooth dg-schemes:</u> $[\mathcal{O}^{\#}_X$ is ~~~~ ∞-mally free$]$

$\leftarrow$ ~~w/o~~ without d

X^0 smooth alg. variety

$$\mathcal{O}^{\#}_X \text{ (Zar. locally)} = S^\bullet_{\mathcal{O}_{X^0}}(E^{\le -1})$$

graded vector bundle

<u>Note:</u> $\pi_0(X)$ does not have to be smooth

$$\operatorname{Hom}_{\text{dg-sch}}(\operatorname{Spec}\mathbb{C}, X) = \operatorname{Hom}_{\text{sch}}(\operatorname{Spec}\mathbb{C}, \pi_0(X))$$

$x \in \pi_0(X)$ $\mathbb{C}$-point $\rightsquigarrow$ $T^\bullet_x X$ $\mathbb{Z}_{\geq 0}$-graded complex (of derivations...)

$H^0 = T_x \pi_0(X)$

$H^i(T^\bullet_x X) =: \pi_{-i}(X, x)$
$i > 0$

$\exists$ Whitehead products: $\pi_i \otimes \pi_j \xrightarrow{[,]} \pi_{i+j-1}$ $\quad \pi_{-1}, \pi_{-2}, \ldots$ $\quad i, j < 0$

<u>Quasiisomorphism</u> $f: X = (X^0, \mathcal{O}^\bullet_X) \to (Y^0, \mathcal{O}^\bullet_Y)$

- isom. of schemes $\pi_0(X) \to \pi_0(Y)$
- $f^* \mathcal{O}^\bullet_Y \to \mathcal{O}_X$ is a q.i.s. ~~of the dg~~...

~~Derived~~

<u>Deformation theory</u>: $T_{[\text{object}]} \underset{\text{moduli space}}{\mathcal{M}} = H^1$ (of a certain sheaf) or H^0

When higher H^i present, $\mathcal{M}$ may be singular

<u>DDT program</u> (derived def. theory) (Drinfeld, Deligne, Konts.): all such $\mathcal{M}$

should come as π_0(some $R\mathcal{M}$) which is smooth dg-scheme

and $H^\bullet T^\bullet_{[\text{object}]} R\mathcal{M}$ = full cohomology of that sheaf

Ciocan-Fontanine, K: $R\,\mathrm{Quot}$, $R\,\mathrm{Hilb}$, $R\mathcal{M}_{g,n}(X,\beta)$

X proj. var. $\subset \mathbb{P}^N$ $\mathcal{F}$ coh. sheaf

$\mathrm{Quot}_{h'}(\mathcal{F}) = \{$ quotients $\mathcal{F} \to \mathcal{G}$ with Hilbert pol. $h_{\mathcal{G}} = h' \}$ mod $\mathrm{Aut}(\mathcal{G})$

$= \{$ subsheaves $\mathcal{K} \subset \mathcal{G}$ with Hilb $= h_{\mathcal{F}} - h' \}$

$\mathrm{Sub}_h(\mathcal{G})$ "Grassmannian"

$$T_{[\mathcal{K}]}\, \mathrm{Sub}_h(\mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{F}/\mathcal{K})$$

Th. $\exists$ smooth dg-scheme $R\,\mathrm{Sub}_h(\mathcal{F})$ with

$\pi_0 = \mathrm{Sub}_h$ $\quad H^\bullet T^\bullet_{[\mathcal{K}]} R\,\mathrm{Sub} = \mathrm{Ext}^\bullet(\mathcal{K}, \mathcal{F}/\mathcal{K})$

When $\mathcal{F} = \mathcal{O}_X$ $\quad \mathrm{Quot}_{h'}(\mathcal{F}) = \mathrm{Hilb}_{h'}(X) =$

$= \{$ subschemes $Z \subset X$ with Hilb. poly $= h \}$

Z smooth l.c.i $\Rightarrow T_{[Z]}\,\mathrm{Hilb} = H^0(Z, N_{Z/X})$

$\forall m$

Th. $\exists$ smooth dg-scheme $R\,\mathrm{Hilb}_h^{\leq m}(X)$ s.t.

$H^i T^\bullet_{[Z]} R\,\mathrm{Hilb}_h^{\leq m} = \mathrm{Ext}^i(\mathbb{L}^\bullet_{Z/X}, \mathcal{O}_Z)$ $\quad i \leq m$

s.t. ~~for~~ a l.c.i Z

$$H^i\, T_{[Z]}\, R\mathrm{Hilb} = H^i(Z, \mathcal{N}_{Z/X}).$$

<u>Important</u>: Many $\mathcal{M}$'s are in fact <u>stacks</u>
with $H^0, H^{-1}(T_{[\text{object}]}\mathcal{M}) = H^1, H^0$ (sheaf of ∞-mal symmetries)

So $R\mathcal{M}$'s should be <u>dg-stacks</u>

<u>Naive def</u>: $(S, \mathcal{O}_S^\bullet, d)$

S: Artin ~~Deligne Mumford~~ stack

$\mathcal{O}_S^\bullet$: sheaf of dg-algebras.

<u>E.g.</u> X / G

X: dg-sch.

G: alg. group.

<u>Example where it suffices</u>: X proj. variety

$\beta \in H_2(X, \mathbb{C})$

$\mathcal{M}_{g,n}(X,\beta)$ Kontsevich moduli space of stable maps

$$(\underset{\text{curve}}{C}, c_1, \dots, c_n) \xrightarrow{f} X \qquad \deg = \beta$$

Deligne-Mumf. stack

$$T_{[C,f,c_i]} = H^0(C, \cancel{f^* T_X}\ \mathrm{Cone}\ T_C(-x_1 \dots -x_n) \to f^* T_Y)$$

Th. (Ciocan-Fontanine, K.) $\exists$ smooth dg-stack $R\mathcal{M}_{g,n}(X,\beta)$

with $H^i(T^\cdot) = H^i(C, -, -)$. $i = 0, 1$.

<u>Example where it does not quite suffice</u>

$\mathrm{Bun}_{r,h}(X) = \{$vect. bundles with rank r
Hilb. polyn. $= h\}$

X ↑ pr. var

Artin stack of locally finite type.

Using RQuot, we can: include any ~~fin~~ bounded part of it into a (smooth) dg-stack of the form RQuot / GL

and these are compatible up to qis.

<u>Lacking</u>: A def. of a dg-stack flexible enough to allow gluing w.r.t. qis

K. Behrend; ~~con~~ a similar more flexible concept
B. Toen of a dg-scheme

? Functors represented by dg-moduli schemes / stacks
On derived category $\{dg\text{-}Sch\}[qis^{-1}]$.

Morphisms not explicit

<u>M. Manetti</u>: For formal germs of moduli schemes.