Analytic complete equivalence relations and their degree spectra

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Let $\mathcal{A}$ be a countable structure in language L and E be an equivalence relation on structures in L.

Question 1. How complicated is $M_E(\mathcal{A}) = \{\mathcal{B} : \mathcal{B} \mathrel{E} \mathcal{A}\}$?

Question 2. How complicated is $I_E(\mathcal{A}) = \{e : \varphi_e = D(\mathcal{B}) \wedge \mathcal{B} \mathrel{E} \mathcal{A}\}$?

$D(\mathcal{B})$ denotes the atomic diagram of $\mathcal{B}$ in the language $L = (R_i)_{i \in I}$,

$$D(\mathcal{B}) = \bigoplus_{i \in I} R_i^{\mathcal{B}}.$$

Question 3. How complicated is the relation E in a specific class o structures?

Formal setting

To answer questions like Question 1 and 3 we consider the following setting:

Let L be a relational language with relation symbols $(R_i/a_i)_{i\in\omega}$, then

$$Mod(L) = \prod_{i\in\omega} 2^{\omega^{a_i}}$$

is a Polish space and we can develop the Borel hierarchy $(\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha)$, projective hierarchy $(\mathbf{\Sigma}^1_\alpha, \mathbf{\Pi}^1_\alpha, \mathbf{\Delta}^1_\alpha)$ in the usual way.

Theorem (Vaught)

A set $S \subseteq Mod(L)$ is $\mathbf{\Sigma}^0_\alpha$ $(\mathbf{\Pi}^0_\alpha)$ if and only if it is definable by a Σ^0_α (Π^0_α) formula in $L_{\omega_1,\omega}$.

Reducibility

Definition

Let E be a binary relation on a Polish space X and F be a binary relation on a Polish space Y, then E is reducible to F if there is a function $f : X \to Y$ such that for all $x_1, x_2 \in X$

$$x_1 \, E \, x_2 \Leftrightarrow f(x_1) \, F \, f(x_2).$$

E is Borel reducible to F, $E \leq_B F$ if f is Borel.

If $X = Mod(L_1)$ and $Y = Mod(L_2)$, then E is computably reducible to F $E \leq_c F$ if there is a Turing operator Φ such that $\Phi^{D(\mathcal{S})} = D(f(\mathcal{S}))$ for $\mathcal{S} \in Mod(L_1)$.

Definition

E is a Γ-complete relation if $E \in \Gamma$ and every relation in Γ is Borel reducible to E.

Examples

Two structures $\mathcal{A}$ and $\mathcal{B}$ are bi-embeddable, $\mathcal{A} \approx \mathcal{B}$ if either is isomorphic to a substructure of the other.

Theorem (Louveau, Rosendal '05)

Bi-embeddability on graphs, $\approx_G$, is a $\mathbf{\Sigma}^1_1$ complete equivalence relation.

Theorem (Calderoni, Thomas '19)

Bi-embeddability on abelian groups, $\approx_A$, is a $\mathbf{\Sigma}^1_1$ complete equivalence relation.

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Theorem (Friedman, Stanley '89;folklore;Hjorth '00)

Isomorphism on graphs $\cong_G$ is

1. *complete among isomorphism on classes of structures,*
2. *not Borel,*
3. *not $\boldsymbol{\Sigma}^1_1$ complete.*

The isomorphism spectrum of a structure, the set of Turing degrees of its isomorphic copies is one of the classic notions studied in computable structure theory.

Fokina, Semukhin, and Turetsky; Montalbán; and Yu independently suggested to study degree spectra with respect to equivalence relations.

Definition

Given an equivalence relation E on $Mod(L)$ and $\mathcal{A} \in Mod(L)$, the degree spectrum of $\mathcal{A}$ w.r.t E is

$$DgSp_E(\mathcal{A}) = \{X \in 2^\omega : \exists \mathcal{B}(\mathcal{B} \mathrel{E} \mathcal{A} \,\&\, D(\mathcal{B}) \equiv_T X)\}$$

Observation: The complexity of the equivalence relation restricts the complexity of its degree spectra.

Proposition (folklore)

If E is $\mathbf{\Pi}^0_\alpha$, then for every $\mathcal{A} \in Mod(L)$ $DgSp_E(\mathcal{A})$ is $\mathbf{\Sigma}^0_{\alpha+1}$.

Let $\mathcal{A} \equiv_n \mathcal{B} \Leftrightarrow Th_n(\mathcal{A}) = Th_n(\mathcal{B})$.

Theorem (Fokina, Semukhin, Turetsky '19)

The class $high_n = \{X : X^{(n)} \geq_T \emptyset^{(n+1)}\}$ is not a $\equiv_n$ spectrum, but it is a $\equiv_{n+1}$ spectrum.

Proof idea. First, show that $high_n$ is not $\mathbf{\Sigma}^0_{n+2}$ using forcing. Notice that $\equiv_n$ is Π^0_{n+1}. Thus, $high_n$ can not be a $\equiv_n$ spectrum by Proposition.

But it is possible to construct a structure $\mathcal{A}$ such that $DgSp_{\equiv_{n+1}}(\mathcal{A}) = high_n$.

Another related and important example arises from Scott's isomorphism theorem:

Proposition (folklore)

Every isomorphism spectrum is Borel.

Degree spectra of Σ^1_1 complete equivalence relations

Fokina, R., and San Mauro '19: Bi-embeddability spectra of structures.

Bi-embeddability does not allow coding.

Theorem (Knight '86)

Let $X \subseteq \omega$. Tfae:

1. *X is c.e. in every isomorphic copy of $\mathcal{A}$.*
2. *X is enumeration reducible to $\exists - tp_{\mathcal{A}}(\bar{a})$ for some $\bar{a} \in A^{<\omega}$.*

Example: Slaman; Wehner '98: There is a structure $\mathcal{A}$ with
$DgSp_{\cong}(\mathcal{A}) = \{X : X >_T \emptyset\}$.
Bouquet graph of Wehner family $\{\{n\} \oplus D : D \text{ finite} \;\&\; W_n \neq D\}$

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Theorem (Fokina, R., San Mauro '19)

There is a graph $\mathcal{G}$ such that $DgSp_{\approx}(\mathcal{G}) = \{\{n\} \oplus D : D \text{ finite } \& \ W_n \neq D\}$

This and similar results are obtainable by using strong codings that include negative information (Csima, Kalimullin '10).

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However, its hard to obtain negative results.

Until now, the only examples of sets that can not be bi-embeddability spectra are sets that are not upwards closed.

Elementary bi-embeddability

Two structures $\mathcal{A}$ and $\mathcal{B}$ are elementary bi-embeddable if either is isomorphic to an elementary substructure of the other.

R. ’18: Elementary bi-embeddability ($\approxeq$) spectra

- Bi-embeddability spectra allow coding: If $\mathcal{A} \preccurlyeq \mathcal{B}$, then for all $\bar{a} \in A^{<\omega}$ $\exists - tp_{\mathcal{A}}(\bar{a}) = \exists - tp_{\mathcal{B}}(\bar{a})$.
- Most examples of isomorphism spectra carry over.
- $\approxeq$-spectra, $\approx$-spectra, and $\cong$-spectra have not been separated.
- The complexity of elementary bi-embeddability and elementary embeddability seems to be poorly understood.

Theorem (R.)

The elementary bi-embeddability relation on graphs is $\boldsymbol{\Sigma}^1_1$-complete.

We prove this theorem by giving a reduction from $\hookrightarrow_G$ to $\preccurlyeq_G$. It then follows from the completeness of $\hookrightarrow_G$ (Louveau, Rosendal) that $\preccurlyeq_G$ is $\boldsymbol{\Sigma}^1_1$ complete. That $\approxeq_G$ is $\boldsymbol{\Sigma}^1_1$ complete is an immediate corollary.

We do a Marker extension using structures with a special model theoretic property to obtain a result about degree spectra.

The complexity of $\approxeq$

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Theorem (R.)

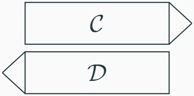
Let $\mathcal{G}$ be a graph, then there exists a graph $\hat{\mathcal{G}}$ such that

$$DgSp_{\approxeq}(\hat{\mathcal{G}}) = \{X : X' \in DgSp_{\approx}(\mathcal{G})\}.$$

Proof sketch

Given $\mathcal{G}$ we first produce a structure $f(\mathcal{G})$ by replacing edges with copies of a L−structure $\mathcal{C}$ and non-edges with copies of $\mathcal{D}$.

$\mathcal{A}: \ a \longrightarrow b$ $\qquad g(\mathcal{A}) : a^g$



b^g

Formally: $f(\mathcal{G})$ is an $L \cup \{V/1, O/3\}$ structures where we have a bijection $f : G \to V$ and the L-reduct of $O(f(a), f(b), -)$ is isomorphic to $\mathcal{C}$ if aEb and $\mathcal{D}$ if $\neg aEb$, no L-symbol holds on elements of V and the sets V, and $O(a, b, -)$ for $a, b \in V$ are pairwise disjoint.

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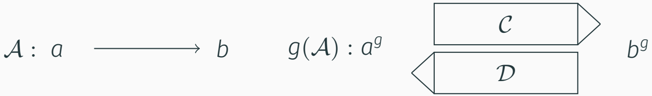

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If $h : \mathcal{G}_1 \hookrightarrow \mathcal{G}_2$, then there is an induced embedding $f(h) : f(\mathcal{G}_1) \hookrightarrow f(\mathcal{G}_2)$. To show that $f(h)$ is elementary we show that player II has a winning strategy in the Ehrenfeucht-Fraïssé games $G_n((f(\mathcal{G}_1), \bar{a}), (f(\mathcal{G}_2), f(h)(\bar{a}))$ for all n, and $\bar{a} \in f(G_1)^{<\omega}$.

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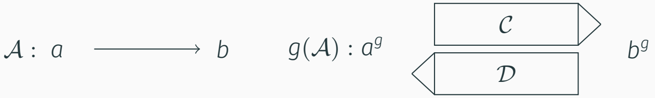

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That $\mathcal{G}_1 \hookrightarrow \mathcal{G}_2$ iff $f(\mathcal{G}_1) \preccurlyeq f(\mathcal{G}_2)$ it is sufficient that

1. $\mathcal{C} \not\cong \mathcal{D}$, $\mathcal{C} \equiv \mathcal{D}$,
2. $\mathcal{C} \not\preccurlyeq \mathcal{D} \wedge \mathcal{D} \not\preccurlyeq \mathcal{C}$.

In particular, $\mathcal{G}_1 \approx \mathcal{G}_2$ iff $f(\mathcal{G}_1) \approx f(\mathcal{G}_2)$. We can code the structures $f(\mathcal{G})$ into a graph using standard codings.

For $DgSp_{\approxeq}(f(\mathcal{G})) = \{X : X' \in DgSp_{\approx}(\mathcal{G})\}$ it is sufficient that

1. for all $\mathcal{A} \approx \mathcal{G}$ $\mathcal{A} \geq_T f(\mathcal{A})$,
2. for all $\mathcal{B} \approxeq f(\mathcal{G})$ there is $\mathcal{A}$
    2.1 with $f(\mathcal{A}) \cong \mathcal{B}$,
    2.2 and $\mathcal{B}' \geq_T \hat{\mathcal{A}} \cong \mathcal{A}$.

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(2)(a) is essential and non-trivial, e.g. take $\mathcal{C} = (\omega, \omega + \zeta)$, $\mathcal{D} = (\omega + \zeta, \omega)$. Then we would get that $f(\mathcal{G})' \geq_T \hat{\mathcal{G}} \cong \mathcal{G}$ but the structure obtained if we use $\mathcal{C} = (\omega, \omega) = \mathcal{D}$ would elementary embed into $f(\mathcal{G})$.

1. $\mathcal{C} \equiv \mathcal{D}$,

2'. for every $\mathcal{A} \ncong \mathcal{C}$, $\mathcal{A} \not\preccurlyeq \mathcal{C}$,

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Minimal Models

Definition

1. A structure $\mathcal{A}$ is minimal, if there is no $\mathcal{B}$ such that $\mathcal{B} \preccurlyeq \mathcal{A}$.

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Question (Vaught): What is the number of minimal models a theory can have?

Theorem (Fuhrken '66)

There is a theory with $2^{\aleph_0}$ minimal models.

Theorem (Shelah '71)

For every $\kappa \leq \aleph_0$, there is a theory with κ minimal models.

For $\nu \in 2^{<\omega}$ define $F_\nu : 2^\omega \to 2^\omega$, $\sigma \mapsto \nu +_2 \sigma$ (where ν is interpreted as $\nu^\frown \bar{0}$ and $+_2$ is base 2 addition).

Let $R_\nu = \{\sigma \in 2^\omega : \nu \preceq \sigma\}$ and consider the theory T of

$$\mathcal{A} = (2^\omega, \langle F_\nu \rangle_{\nu \in 2^{<\omega}}, \langle R_\nu \rangle_{\nu \in 2^{<\omega}}).$$

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Shelah used T and variations of T to prove his theorem. It is easy to see that

1. T has quantifier elimination,
2. the substructure $\langle \sigma \rangle$ generated by $\sigma \in 2^\omega$ is an elementary substructure of $\mathcal{A}$,
3. $\langle \sigma \rangle$ is minimal,
4. if $\exists^\infty i\ \sigma(i) \neq \tau(i)$, then there is a Σ^c_2 sentence distuingishing $\langle \sigma \rangle$ and $\langle \tau \rangle$.

$$\exists x \bigwedge_{\nu \preceq \sigma} R_\sigma(x)$$

Lemma

Let X be $\Delta^0_2(Y)$ for a set Y, then there exists a sequence of structures $(\mathcal{C}_i)_{i\in\omega}$, uniformly computable in Y, such that

$$\mathcal{C}_i \cong \begin{cases} \langle\bar{0}\rangle & \text{if } i \in X, \\ \langle\bar{1}\rangle & \text{if } i \notin X. \end{cases}$$

We do a Marker extension with $\langle\bar{0}\rangle$ and $\langle\bar{1}\rangle$ to obtain the result that for every graph $\mathcal{G}$, there is a graph $\hat{\mathcal{G}}$ such that

$$DgSp_{\cong}(\hat{\mathcal{G}}) = \{X : X' \in DgSp_{\approx}(\mathcal{G})\}.$$

Ending Thoughts

- We still do not know how to seperate isomorphism and bi-embeddability spectra.
- The main result can be used to obtain the first "non-trivial" example of a set of degrees that can not be a bi-embeddability spectrum.

Corollary

Let $X, Y >_T \emptyset'$ and $X \not\equiv_T Y$, then

$$\{Z : Z' \geq_T X\} \cup \{Z : Z' \geq_T Y\}$$

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Question: Let $\mathfrak{F}$ be a $\approx$, $\cong$, or $\equiv$ spectrum, is $\{X' : X \in \mathfrak{F}\}$?

Question: Examples of upwards closed sets of Turing degrees that are $\mathbf{\Sigma}^1_1$ and not Borel?

Thank you!